Tightness of Hypergraphs

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Some terminology

- A **hypergraph** H = set $V(H)$ of **vertices** $+$ a set $E(H)$ of **edges** which are subsets of V.
- If $\forall\ e \in E(H)\ |e| = k$ then H is a k**-graph**. A **graph** is a 2-graph.
- A k**-coloring** of a hypergraph is a vertex coloring onto the set of k colors.
- A k-coloring of a hypergraph **divides** an edge if all the vertices are differently colored and **separates** it, if the coloring does not divide any other edge

k-trees and k-forests

- A graph is connected if and only if any 2-coloring of it divides some edge.
- We say that a k-graph is TIGHT if any k-coloring of it divides some edge.
- A k**-tree** is a critical in edges tight k-graph.
- A k**-forest** is a k-graph such that any edge is separable by some k coloring.
- A k-graph is a k-tree if and only if it is a tight k-forest.

Bounds

- If $k \geq 3$ then there are k-trees with the same number of vertices and different numbers of edges
- Due to Lövasz it is known that the maximal size of a k-tree with n vertices is $\binom{n-1}{k-1}$
- It can be shown that any k-tree with n vertices has at least $\lceil \frac{2}{n-k+2}\binom{n}{k} \rceil$ edges.
- It is conjectured that this bound is sharp.

Neighborhoods

- Given a 3-graph $G = (V, E)$ and $v \in V$. The **trace** of v is the graph defined on the vertex set $V \setminus v$ and there is an edge $\{a, b\}$ if the triple $\{a, b, v\} \in E$.
- A 3-graph with $|V| \equiv 0, 2 \pmod 3$ is locally a tree if the traces of all vertices are trees.
- A 3-graph with $|V| \equiv 1 \pmod 3$ is locally a tree if the trace of one vertex is unicyclic and the trace of any other vertex is a tree.
- It is not difficult to show that the conjecture is equivalent to the following:*There are tight 3-graphs which are locally trees*

Observation

- The matter of Ramsey Theory is finding the maximal number of colors for which we can assure that any vertex coloring of a hypergraph has a monochromatic edge.
- The matter of Tightness is finding the minimal number of colors for which we can assure that any vertex coloring of a hypergraph has a heterochromatic edge.
- So, tightness of hypergraphs is in essence an Anti-ramsey property.

Sterboul's work

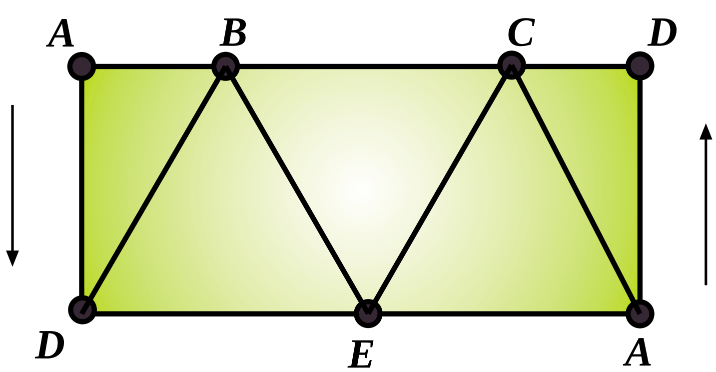
- F.Sterboul, *A problem on triples*, Discrete Mathematics, 17, 1977, 191-198.
  Here he shows that the conjecture is true for $3 \leq n \leq 12$.
- F.Sterboul, *A problem on constructive combinatorics and related questions*, Colloquia Mathematica János Bolyai, 18, 1976.
  Here he claims a proof of the conjecture for $n \equiv 0, 2 \pmod 3$.
- Actually he succeeded in the construction of 3-graphs which are locally trees. Unfortunately, the proofs that those 3-graphs are tight are incorrect and we could not find an easy way to correct them.

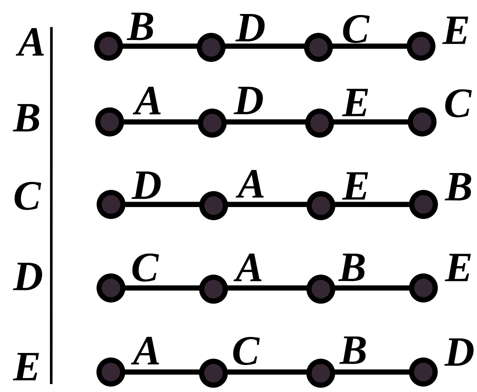
The equation x + y = z

- Arocha J., Bracho J., Neumann-Lara V. *On the minimum size of tight hypergraphs*, Journal of Graph Theory, 16 -4 , 1992, 319-326.
- For any prime number $p \geq 7$ and for any 3-coloring of the elements of $\mathcal{Z}_p^*$ there is a heterochromatic solution of the equation $x + y = z \pmod{p}$.
- For any $n = \frac{p-1}{2}$ where p is a prime number the conjecture is true.

3-chains and 3-cycles

- A **3-chain** is a 3-graph such that the traces of all the vertices are chains.
- A **3-cycle** is a 3-graph such that the traces of all the vertices are cycles.
- Of course, any 3-chain is locally a tree.





- Observe that this is a triangular embedding of $\mathbf{K}_5$ in the Möebius band.

Embeddings of complete graphs

- 3-cycles are triangular embeddings of complete graphs in closed surfaces.
- 3-chains are triangular embeddings of complete graphs in surfaces with boundary.
- Thanks to the efforts of Ringel, Young and others to solve the Heawood conjecture we know that 3-cycles exist for any possible n.
- A 3-chain can be obtained by deleting a vertex from a 3-cycle.
- **Are all 3-chains tight?**

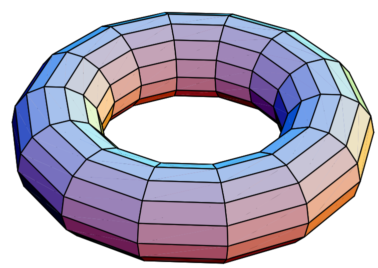
The coupling construction

- Arocha J., Bracho J., Neumann-Lara V. *Tight and Untight Triangulation of Surfaces by Complete Graphs*, JCT (B), 63-2 , 1995, 185-199.
- A 3-cycle $2G$ of order $2n$ can be constructed given a 3-chain G of order n. The 3-cycle $2G$ is tight if and only if G is tight and has connected boundary.
- The smaller number for which there exist an untight 3-cycle is 16.
- Tightness invariants differenciate complete graph embeddings in surfaces.
- All known examples of untight 3-cycles and 3-chains are non orientable.

The Skolem 3-chains

- Consider the abelian group $Z_3 \oplus Z_t$ for t odd as a vertex set. This set can be arranged in a table which in fact is a torus.

00	***01***	***02***	***...***	***0(t-1)***
10	***11***	***12***	***...***	***1(t-1)***
20	***21***	***22***	***...***	***2(t-1)***



- Describe a set of triples as the union of the three following sets:

  (a) The set of columns of the table.

  (b) Given two vertices ab and ac in a same row, take the third $\left(a+1, \frac{b+c}{2}\right) \pmod{Z_3 \oplus Z_t}$

  (c) Given two vertices ab and ac in a same row, take the third $\left(a+1, \frac{b+c+1}{2}\right) \pmod{Z_3 \oplus Z_t}$

Properties of Skolem 3-chains

- The 3-graphs defined by the first two sets of triples are the Skolem Steiner Triple Systems.
- Denote by Sk_t the 3-graph defined by the three sets of vertices.
  (a) Sk_t is a non orientable 3-chain
  (b) The group $Z_3 \oplus Z_t$ acts transitively in the 3-graph Sk_t
  (c) Sk_t is tight !!
- So, the conjecture on the minimum size of tight hypergraphs is proved for $n \equiv 3 \pmod 6$

The case $n \equiv 0 \pmod 6$

- The definition of Skolem 3-chains depends on the possibility of the division by 2 in Z_t. If t is even this is not possible.
- However, if we replace the division by 2 with the following operation
$$\left[\frac{x}{2}\right] = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x+t-1}{2} & \text{if } x \text{ is odd} \end{cases}$$
then the 3-graphs Sk_t are well defined for t even.
- The 3-graphs Sk_t for t even are also vertex transitive non orientable tight 3-chains
- Thus, the conjecture on the minimum size of tight hypergraph is proved for $n \equiv 0 \pmod 6$

The case $n \equiv 4 \pmod 6$

- Consider a 3-chain G of order $3k$ and suppose that it has k disjoint triplets.
- If the boundary is connected then glueing a disk (triangulated with a new vertex) to the boundary and deleting the disjoint triplets we obtain a new 3-chain G' of order $3k+1$.
- If the boundary is not connected then this operation can be done only if a hamiltonian cycle with special properties can be found in the graph formed by the boundary edges and the edges of the disjoint triples.
- This is possible for Skolem's 3-chains Sk_t with t odd. The proof of the tightness of Sk'_t is based on the tightness of Sk_t.

The open cases

- The cases that remain open are

$$n \equiv 1 \pmod 6 \text{ and } n \equiv 2 \pmod 3$$

Tight and Untight Equations

- An equation $ax + by + cz = 0$ over Z_p is called tight if for any 3-coloring of Z_p there is an heterochromatic solution of it.
- (B.Llano. PhD Thesis, In preparation) The following is the complete list of untight equations:

(a) $x + y + z = 0, \quad x + y + 2z = 0$

(b) $x + y = \frac{1 \pm \sqrt{5}}{2} z$ and $p \equiv \pm 1 \pmod{10}$

(c) $x + y = 2z$ and $\langle \widehat{2} \rangle \neq Z_p^*/\{1, -1\}$.

(d) $x + by = b^2 z$ and $\{1, b, b^2\}$ is a subgroup of Z_p^*.

(e) $x + by + (b+1)z = 0$ and $b, -(b+1)$ are quadratic residues modulo $p \equiv 1 \pmod{4}$.

Coloring the plane

- Abrego B., Arocha J., Fernandez S. and Neumann-Lara V. , *Tight sets of triangles in* R^2, Preprint IMATE-UNAM, 445, 7 Dec. 1995.
- Here we study the sets of triangles in the plane having the property of tightness.
- A typical theorem in this work is the following
- For any positive real number k and any 3-coloring of the plane an isosceles tricolored triangle having a side of lenght k can be found.

Problems

- A 3-forest is called **saturated** if after the addition of any triple it becomes a non 3-forest.
- Are all saturated 3-forests tight?
- Are all orientable 3-cycles tight?
- Is the set of all triangles in the plane having fixed area tight?