

Manifolds arising from configurations of points in the projective space

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Preliminaries

\mathbb{P}^2 = the real projective plane

$\text{Pr}^2 = GL(3, \mathbb{R})/\mathbb{R}^* = \left\{ \begin{array}{l} \text{the group of all projective} \\ \text{isomorphisms} \end{array} \right.$

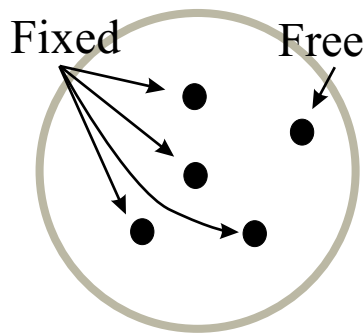
$(\mathbb{P}^2)^5 = \{(p_1, \dots, p_5) : p_i \in \mathbb{P}^2\} = \left\{ \begin{array}{l} \text{the space of all} \\ \text{5-tuples of points} \end{array} \right.$

$\mathbb{P}_5^2 = (\mathbb{P}^2)^5 / \text{Pr}^2 = \left\{ \begin{array}{l} \text{the space of all configurations} \\ \text{of five points} \end{array} \right.$

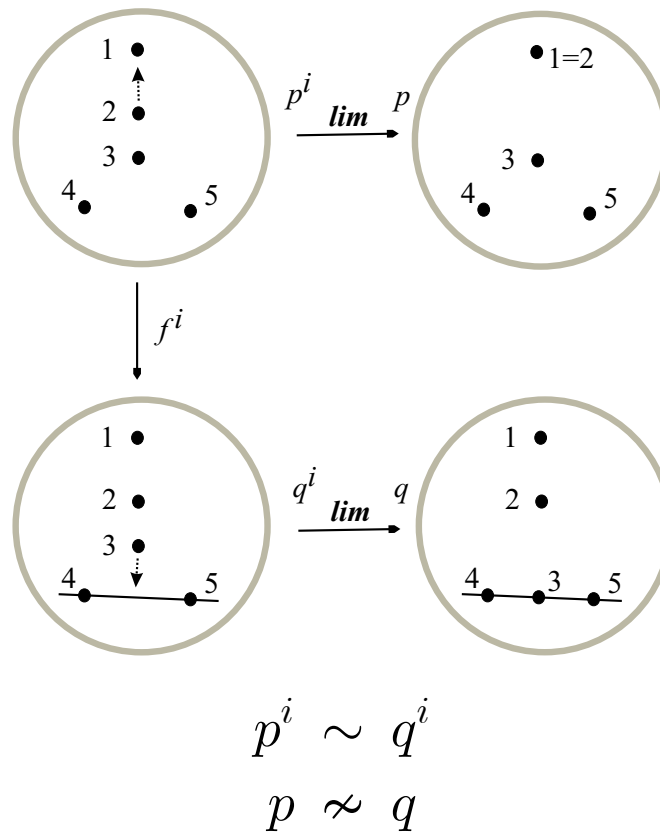
$((p_1, \dots, p_5) \sim (q_1, \dots, q_5)) \Leftrightarrow (\exists f \in \text{Pr}^2 : f(p_i) = q_i)$

Two properties

- \mathbb{P}_5^2 is compact
- $\dim \mathbb{P}_5^2 = 2$



\mathbb{P}_5^2 is not Hausdorff



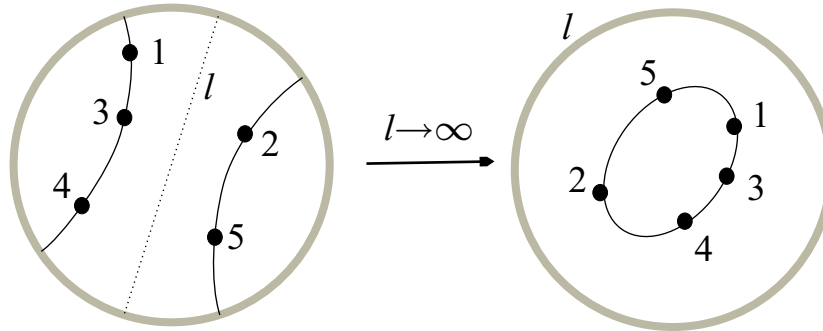
We want to obtain a subspace L of \mathbb{P}_5^2 such that L is a compact surface.

If L is too large, then it will be non Hausdorff. If L is too small, then it will be non compact.

An example

Let L be the subspace of \mathbb{P}_5^2 of all configurations of **five different points no four in a line**.

If the five points are in general position then there is exactly one projective conic passing through them.

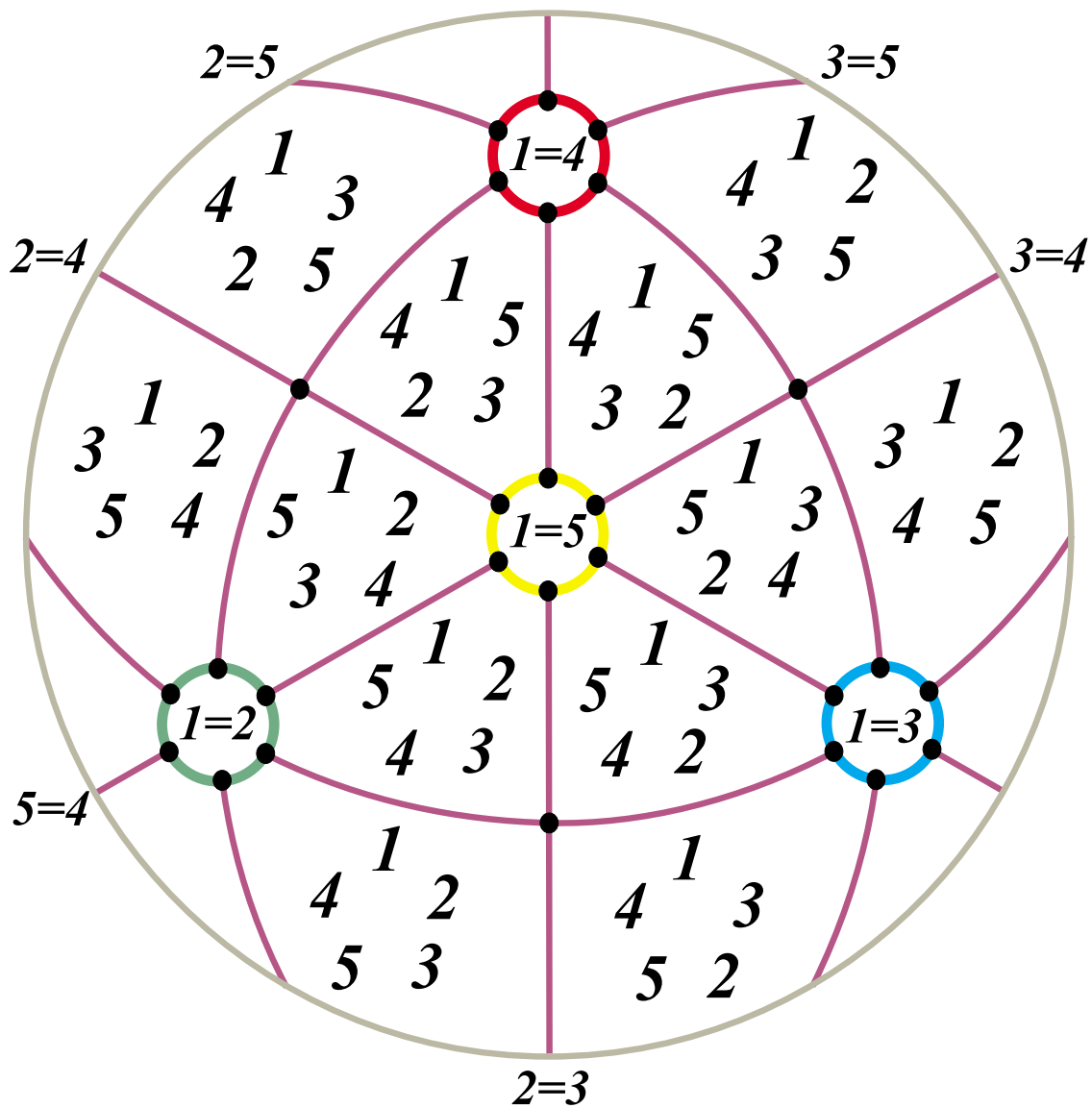


Configuration of 5 points in general position are classified by dihedral permutations of 5 numbers. Two dihedral permutations are adjacent iff one can be obtained from the other by swapping two numbers.

$$\begin{array}{c} 4 & 1 & 2 \\ & & \searrow \\ & 5 & 3 \end{array} \longleftrightarrow \begin{array}{c} 4 & 1 & 3 \\ & & \searrow \\ & 5 & 2 \end{array}$$

Therefore, the space L can be obtained taking twelve pentagons (with interiors) one for each dihedral permutation and gluing them by the rule of swapping adjacent numbers. If we do this, then we obtain the surface $5\mathbb{P}^2$ the connected sum of five projective planes.

Five different points no four in a line



The surface $5\mathbb{P}^2$ (in all circles antipodal points must be identified)

The general problem

$\mathbb{P}_k^n = (\mathbb{P}^n)^k / \text{Pr}^n = \left\{ \begin{array}{l} \text{the space of all configurations} \\ \text{of } k \text{ points in } \mathbb{P}^n \end{array} \right.$

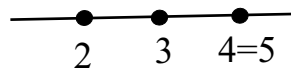
A **type** A_α is a subset of indices $A \subseteq \{1, \dots, k\}$ with a number $\alpha \in \{1, \dots, n\}$.

A configuration $[p_1, \dots, p_k] \in \mathbb{P}_k^n$ has type A_α iff

$$\dim \langle p_i : i \in A \rangle \leq \alpha$$

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Example: The configuration has types



$$A_0 \quad \forall A \quad |A| \leq 1; \{4, 5\}_0$$

$$A_1 \quad \forall A \quad |A| \leq 2; A_1 \quad \forall A \subseteq \{2, 3, 4, 5\}$$

Let \mathcal{R} be a partition of all types into **permitted** types and **prohibited** types. Denote by $\mathbb{P}_k^n(\mathcal{R})$ the space of all configurations $\mathbf{p} \in \mathbb{P}_k^n$ such that any type of \mathbf{p} is permitted..

Problem: For which \mathcal{R} the space $\mathbb{P}_k^n(\mathcal{R})$ is a compact manifold?

The solution: points in the projective line.

Theorem. $\mathbb{P}_k^1(\mathcal{R})$ is a compact manifold if and only if

1. $\forall A \quad |A| \leq 1 \Rightarrow A_0$ is permitted;
2. $(A_0$ is permitted and $B \subseteq A) \Rightarrow B_0$ is permitted
3. \forall partition $A \dot{\cup} B = \{1, \dots, k\}$
 $(A_0$ is permitted) $\Leftrightarrow (B_0$ is prohibited)

The solution: points in the projective plane.

Theorem. $\mathbb{P}_k^2(\mathcal{R})$ is a compact manifold if and only if

1. $\forall A \quad |A| \leq 1 \Rightarrow A_0$ is permitted;
2. $(A_0$ is permitted and $B \subseteq A) \Rightarrow B_0$ is permitted
3. \forall partition $A \dot{\cup} B = \{1, \dots, k\}$
 $(A_0$ is permitted) $\Leftrightarrow (B_1$ is prohibited)
4. \forall partition $A \dot{\cup} B \dot{\cup} C = \{1, \dots, k\}$
 $(A_0, B_0$ are permitted) $\Leftrightarrow (C_0$ is prohibited)
 $(A_0, B_0$ are prohibited) $\Leftrightarrow (C_0$ is permitted)

A final example: 5 lines in the affine plane.

Let us consider the topological space L of configurations of 5 lines in the affine real plane \mathbb{A}^2 such that:

- No 3 of them are parallel.
- No 3 of them are equal.
- No all are concurrent.

Adding the line at the infinity and using the polarity between points and lines in the projective plane \mathbb{P}^2 we obtain:

The space L is homeomorphic to the space of configurations of 6 points in the real projective plane \mathbb{P}^2 such that:

- No 5 of them are in a line
- No 4 of them containing p_6 are in a line.
- No 3 of them are equal.
- No other point is equal to p_6 .

It is easy to check the conditions in the preceding theorem thus obtaining:

The space L is a compact 4-dimensional manifold.