# On the Minimum Size of Tight Hypergraphs

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## ABSTRACT

A $k$-graph, $H = (V, E)$, is *tight* if for every surjective mapping $f\colon V \to \{1, \dots, k\}$ there exists an edge $\alpha \in E$ such that $f|_\alpha$ is injective. Clearly, 2-graphs are tight if and only if they are connected. Bounds for the minimum number $\varphi_n^k$ of edges in a tight $k$-graph with $n$ vertices are given. We conjecture that $\varphi_n^3 = \lceil n(n-2)/3 \rceil$ for every $n$ and prove the equality when $2n + 1$ is prime. From the examples, minimal embeddings of complete graphs into surfaces follow. © 1992 John Wiley & Sons, Inc.

## 1. INTRODUCTION

The concepts of heterocoloring and heterochromatic number of hypergraphs are introduced in this paper. They are inspired by the triangle-free disconnection of a digraph, introduced in [6]. From them, a new type of connectivity-related notion is obtained.

By a t-*coloring* of the hypergraph $H = (V, E)$, we mean a *surjective* mapping from the vertex set $V$ onto a $t$-element set. A $t$-coloring $f$ of $H$ *separates* the edge $\alpha \in E$ if the images by $f$ of the vertices in $\alpha$ are all different. We call $f$ *heterochromatic* (or a t-*heterocoloring* of $H$) if $f$ separates some edge of $H$. The heterochromatic number of $H$, denoted $hc(H)$, is the maximum $t$ for which there exists a $(t-1)$-coloring that is not heterochromatic.

Observe that $hc(H) \leq n + 1$, where $n$ denotes the number of vertices. On the other hand, assuming $E \neq \emptyset$, we have $hc(H) \geq \min\{\alpha; \alpha \in E\}$, and $hc(H)$ is the minimum number $t$ for which any $t$-coloring of $H$ is heterochromatic. Note also that if $H'$ is a spanning subhypergraph of $H$, then $hc(H') \geq hc(H)$.

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A k-*graph,* also called k-*uniform hypergraph,* is a hypergraph for which all edges have exactly $k$ vertices.

A 2-graph is simply a graph. Clearly, for a graph $G$, $hc(G) = c + 1$, where $c$ is the number of connected components of $G$. Hence, $G$ is connected iff $hc(G) \leq 2$.

A $k$-graph is called *tight* iff $hc(H) \leq k$, that is, if it has less than $k$ vertices or if $hc(H) = k$. In a tight $k$-graph, one can find a heterochromatic edge for any $k$-coloring (see Figure 1 for small examples with $k = 3$, where the edges are the shaded triangles).

A tight $k$-graph $H = (V, E)$ is a k-*tree* if for any edge $\alpha \in E$ the $k$-graph $H \backslash \alpha = (V, E \backslash \{\alpha\})$ is not tight.

For $k \geq 3$, $k$-trees on the same vertex set may have different *size* (i.e., number of edges). Take, for example, the 3-graphs on 6 vertices with edge sets

$$\{123, 124, 125, 134, 136, 145, 146, 235, 236, 256\}$$

and

$$\{123, 134, 145, 156, 246, 256, 235, 346\}.$$

Graham and Lovász [5] defined a k-*forest* as a $k$-graph such that any edge is separated by some coloring that does not separate any other edge. Clearly, a

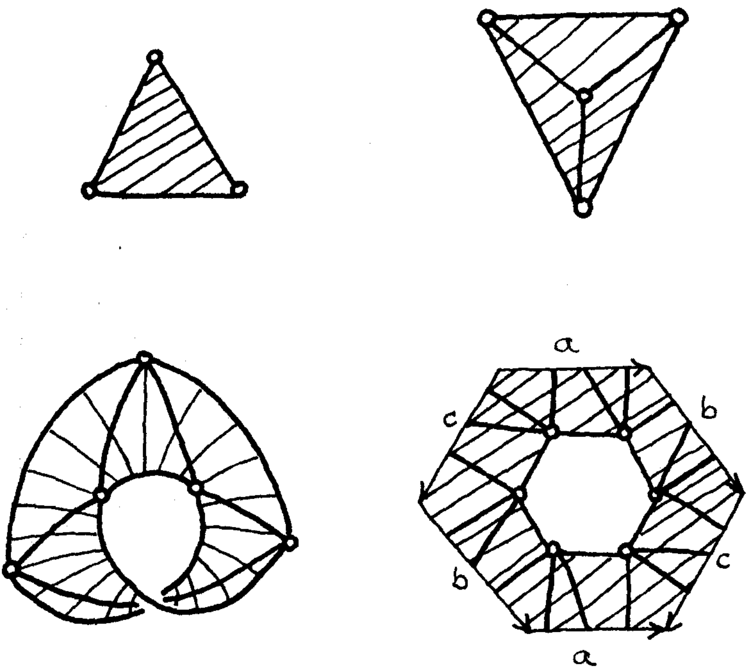


FIGURE 1. Minimal 3-trees.

$k$-graph is a $k$-tree if and only if it is a tight $k$-forest. In [5], Lovász proved that the maximum size of a $k$-forest on $n$ vertices is $\binom{n-1}{k-1}$. Since the $k$-graph whose edges are all $k$-tuples containing a fixed vertex is a $k$-tree, then the maximum size of a $k$-tree on $n$ vertices is $\binom{n-1}{k-1}$.

In this paper we study the minimum size $\varphi_n^k$ of a $k$-tree with $n$ vertices. A $k$-tree with this number of edges will be called a *minimal* $k$-tree. It is easy to see that only for $k = 2$ $k$-trees and minimal $k$-trees coincide, being the usual trees (with $\varphi_n^2 = n - 1$). Examples of minimal 3-trees of order $n \leq 6$ are those of Figure 1.

In Section 2, some properties of tight $k$-graphs are given and $\varphi_n^k$ is bounded below (Corollary 2.6). We conjecture that this lower bound is best possible. In Section 3, restricting to $k = 3$, we prove the conjecture asymptotically with an explicit construction of 3-trees of order $n$ when $2n + 1$ is prime. They are related (Section 4) to triangular embeddings of complete graphs into surfaces.

## 2. THE LOWER BOUND

Let $H = (V, E)$ be a $k$-graph, and let $X$ be a nonempty subset of $V$. Define the *trace* of $X$ as the $(k - 1)$-graph $\mathcal{T}_H(X) = (V \backslash X, E_X)$, where

$$E_X = \{\{v_1, \ldots, v_{k-1}\} \subseteq V \backslash X \mid \exists\, x \in X, \{v_1, \ldots, v_{k-1}, x\} \in E\},$$

and the *skeleton* of $H$ as the $(k - 1)$-graph $\mathcal{S}k(H) = (V, S)$, where

$$S = \{\{v_1, \ldots, v_{k-1}\} \subseteq V \mid \exists\, x \in V, \{v_1, \ldots, v_{k-1}, x\} \in E\},$$

**2.1. Basic Lemma.** $H = (V, E)$ is a tight $k$-graph if and only if for each nonvoid subset $X$ of $V$, $\mathcal{T}_H(X)$ is a tight $(k - 1)$-graph. (Furthermore, it is sufficient to consider sets $X$ of cardinality at most $\lfloor n/k \rfloor$).

***Proof.*** Suppose $H$ is not tight, and that we are in the nontrivial case, i.e., $n = |V| \geq k$. Then there is a $k$-coloring $f$: $V \to \{1, \ldots, k\}$ that is not heterochromatic. Set $X = f^{-1}(k)$ (we may assume that $|X| \leq \lfloor n/k \rfloor$). Clearly, the restriction of $f$ to $V \backslash X$ is not a heterocoloring of $\mathcal{T}_H(X)$ and thus $\mathcal{T}_H(X)$ is not tight.

On the other hand, suppose there is a subset $X$ of $V$ such that $\mathcal{T}_H(X)$ is not tight. Let $f$: $V \backslash X \to \{1, \ldots, k - 1\}$ be a mapping that is not a heterocoloring of $\mathcal{T}_H(X)$. Painting the vertices of $X$ with color $k$ we obtain a $k$-coloring that is not heterochromatic. ∎

Assume henceforth that $n \geq k$.

**2.2. Proposition.** The skeleton of a tight $k$-graph is the complete $(k - 1)$-graph with $\binom{|V|}{k-1}$ edges.

***Proof.*** For any $(k-1)$-subset of $V$, consider its complement and apply the basic lemma. ∎

Let $\rho: V \to V'$ be a surjective mapping from the vertex set $V$ of the $k$-graph $H = (V, E)$ onto the set $V'$. The $k$-graph $H' = (V', E')$, with

$$E' = \{\{v_1', \ldots, v_k'\} \mid \exists \{v_1, \ldots, v_k\} \in E,\ \rho(v_i) = v_i',\ i = 1, \ldots, k\},$$

is called the *quotient* of $H$ by $\rho$, and will be denoted as $H/\rho$.

**2.3. Proposition.** Quotients of tight $k$-graphs are tight.

***Proof.*** If $f$ is a coloring of $H/\rho$, then $f \circ \rho$ is a coloring of $H$. Since $H$ is tight, there is an edge $\{v_1, \ldots, v_k\}$ separated by $f \circ \rho$ and then $\{\rho(v_1), \ldots, \rho(v_k)\}$ is separated by $f$. ∎

**2.4. Corollary.** $\varphi_n^k \le \varphi_{n+1}^k$.

***Proof.*** Take a minimal $k$-tree on $n + 1$ vertices, identify two vertices, and apply Proposition 2.3. ∎

For $v \in V$, denote by $\mathcal{V}al(v)$ the number of edges in $\mathcal{T}r(v)$.

**2.5. Proposition.** $\varphi_n^k \ge (n/k)\varphi_{n-1}^{k-1}$.

***Proof.*** Let $H = (V, E)$ be a minimal $k$-tree on the $n$-vertex set $V$. For any $v$ in $V$ the trace $\mathcal{T}r(v)$ must be tight. Hence, $\mathcal{V}al(v) \ge \varphi_{n-1}^{k-1}$, and therefore,

$$k|E| = \sum_{v \in V} \mathcal{V}al(v) \ge n\varphi_{n-1}^{k-1}. \quad ∎$$

**2.6. Corollary.**

$$\varphi_n^k \ge \left\lceil \frac{2}{n-k+2}\binom{n}{k} \right\rceil.$$

***Proof.*** Iterate the bound in 2.5 having in view that $\varphi_n^2 = n - 1$. ∎

From now on we concentrate on the case $k = 3$ and for simplicity we denote $\varphi_n = \varphi_n^3$.

**2.7. Proposition.**

$$\left\lceil \frac{n(n-2)}{3} \right\rceil \le \varphi_n \le \binom{n-1}{2}.$$

*Proof.* Just observe 2.6 and Lovász's upper bound. ∎

**2.8. Conjecture.** $\varphi_n = \lceil n(n-2)/3 \rceil$.

For infinitely many $n$, this conjecture is true (Corollary 3.4, below), whereas for general $k$-graphs, the corresponding conjecture is that Corollary 2.6 is in fact an equality.

According to the basic lemma (2.1), this conjecture states that for every minimal 3-tree either all one-vertex traces are trees (case $n \equiv 0, 2 \pmod 3$), or one of them is a unicycle and the rest trees (case $n \equiv 1 \pmod 3$).

For examples of nonisomorphic minimal 3-trees of order 8, consider the 3-graphs $H_1$ and $H_2$, whose set of vertices is $\mathbb{Z}_8$ and whose sets of edges are respectively:

$$V_1 = \{013, 014, 027, 037, 045, 046, 124, 125, 156, 157, 235, 236, 267, 346, 347, 457\}$$

$$V_2 = \{014, 025, 035, 036, 037, 045, 125, 136, 146, 147, 156, 236, 247, 257, 267, 347\}.$$

They were constructed so that the function $(x \to x + 1)$ is an automorphism. The traces of 0 are, respectively, the trees $T_1$ and $T_2$ of Figure 2. Thus, they are nonisomorphic, and from their symmetry it is easy to see that both $H_1$ and $H_2$ are tight.

However, not all trees may appear as one-vertex traces of minimal 3-trees (see [1]).

## 3. AN INFINITE FAMILY OF MINIMAL 3-TREES

For the rest of this paper $p$ will denote a prime number and $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ the field with $p$ elements. Let $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$ be the multiplicative group of $\mathbb{Z}_p$.

**3.1. Proposition.** For any 3-coloring of $\mathbb{Z}_p^*$ there is a solution of the equation $x + y = z$ with different colors.

*Proof.* Suppose that the proposition is false. Then there exists a partition $\pi = \{A, B, C\}$ of $\mathbb{Z}_p^*$ into nonempty blocks such that

$$\begin{Bmatrix} (A+B) \cap C = \varnothing \\ (C+B) \cap A = \varnothing \\ (A+C) \cap B = \varnothing \end{Bmatrix}. \qquad (*)$$

Note that if the partition $\pi = \{A, B, C\}$ satisfies $(*)$ then for any $x$ in $\mathbb{Z}_p^*$, $x\pi = \{xA, xB, xC\}$ is also a partition that satisfies $(*)$. We may suppose

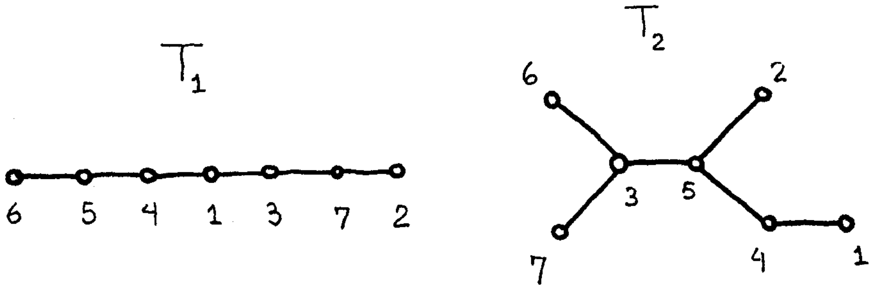


FIGURE 2. Traces of 0 in $H_1$ and $H_2$.

that $|A| \leq |B|$ and $|A| \leq |C|$. If $1 \notin A$ then for any $a \in A$ the new partition $a^{-1}\pi = \{a^{-1}A, a^{-1}B, a^{-1}C\}$ has 1 in the smallest block. So, assume that 1 is in $A$. Let $t$ be the greatest number such that $\{1, \ldots, t\} \subseteq A$. We have $t \geq 1$ and we may consider $t + 1$ in $B$.

**Claim.** If $c \in C$ then $\{c - 1, c - 2, \ldots, c - t\} \subseteq A$.

Indeed, let $C = \{c_1 < c_2 < \cdots\}$ and $h \in \{1, \ldots, t\}$. If $c_i - h \in B$, then $c_i = (c_i - h) + h$, which contradicts (*). This proves the claim for $c_1$, since it has no other alternative. Now, suppose that the claim is true up to $c_{i-1}$. If $c_i - h \in C$, then $c_i = c_j + h$ for some $j < i$. Since $x = t + 1 - h \in \{1, \ldots, t\}$, we have by induction that $c_j - x = c_i - (t + 1) \in A$. But, since $t + 1 \in B$, this contradicts (*). Therefore $c_i - h \in A$, proving the claim.

From the claim, the map $c \mapsto c - 1$ is an injection from $C$ into $A$. But it is not a bijection ($2 \notin C$ and $1 \in A$), contradicting the fact that $A$ is the smallest block. ∎

Observe that Proposition 3.1 cannot be generalized directly to 4-colorings. Indeed, consider the 4-coloring $\{\{1\}, \{-1\}, A, B\}$, where $B = \mathbb{Z}_p^* \backslash A \backslash \{1, -1\}$ and

$$A = \begin{cases} \{2, 5, 6, 9, 10, \ldots, p - 4, p - 3\}, & \text{if } p \equiv 1 \bmod 4; \\ \{2, 3, 6, 7, \ldots, p - 5, p - 4\}, & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

We have $A = -B$, and $x + y = 2$ has no solution with $x \in A$ and $y \in B$. Hence, there is no solution of $x + y + z = w$ with different colors. Note also that Proposition 3.1 is, in a sense, the anti-Ramsey version of a Theorem of Schur (see [8] and [3]).

Denote by $\mathcal{B}_p = (\mathbb{Z}_p^*, E)$ the 3-graph whose edges are the 3-sets $\{x, y, z\}$ such that $x + y = z$. By Proposition 3.1, $\mathcal{B}_p$ is tight. Let $G$ be a subgroup of $\mathbb{Z}_p^*$, and let *nat*: $\mathbb{Z}_p^* \to \mathbb{Z}_p^*/G$ be the canonical mapping. Denote by $\mathcal{B}_p/G$ the quotient of $\mathcal{B}_p$ by *nat*. By Proposition 2.3, $\mathcal{B}_p/G$ is tight too. It is easy to see that the group of automorphisms of $\mathcal{B}_p/G$ is transitive.

Set $\mathcal{L}_p = (V_p, E_p) = \mathcal{B}_p/\{1, -1\}$. Thus $V_p = \{1, 2, \ldots, (p - 1)/2\}$.

**3.2. Lemma.** For any vertex $x$ of $\mathcal{L}_p$, $\mathcal{T}r(x)$ is a path.

***Proof.*** The path $2, 3, \ldots, (p-1)/2$ is the trace of 1. For any vertex $x$ of $\mathcal{L}_p$ there is an automorphism that maps 1 to $x$. Such an automorphism must be an isomorphism between $\mathcal{T}r(1)$ and $\mathcal{T}r(x)$. ∎

**3.3 Theorem.** $\mathcal{L}_p$ is a minimal $k$-tree.

***Proof.*** Recall that $\varphi_n \geq n(n-2)/3$. Let $n = (p-1)/2$, and denote by $w$ the number of edges in $\mathcal{L}_p$. By Lemma 3.2, $\mathcal{V}al(x) = n - 2$ for any vertex $x$ of $\mathcal{L}_p$. Hence

$$3w = \sum_{x \in V_p} \mathcal{V}al(x) = n(n-2).$$

Since $\mathcal{L}_p$ is tight the theorem follows. ∎

**3.4. Corollary.** If $2n + 1$ is a prime number, then $\varphi_n = n(n-2)/3$. ∎

**3.5. Proposition.** $\varphi_n/(n(n-2)/3) = 1 + O(n^{-19/21})$.

***Proof.*** Denote by $\mathbf{p}(x)$ the minimum prime number such that $\mathbf{p}(x) \geq x$ and let $\Theta(n) = n(n-2)/3$. By (3.4), (2.4), and (2.7), we have

$$\Theta(n) \leq \varphi_n \leq \Theta\left(\frac{\mathbf{p}(2n+1) - 1}{2}\right).$$

Let $\alpha = 23/42$. We know (see [14]) that $\mathbf{p}(2n+1) \leq (2n+1)^\alpha + 2n + 1 \leq 2C_1 n^\alpha + 2n + 1$ for sufficiently large $2n + 1$. Therefore

$$1 \leq \varphi_n/\Theta(n) \leq \Theta(C_1 n^\alpha + n)/\Theta(n) \leq C_2 n^{2(\alpha - 1)} + 1 .$$ ∎

## 4. RELATION TO SURFACES

A natural source of candidates for minimal 3-trees comes from minimal embeddings of complete graphs into closed surfaces. Indeed, suppose that we are given a triangular embedding $K_{n+1} \to S$ (or $(K_{n+1} - uw) \to S$ when $n \equiv 1 \bmod(3)$ and $uw$ is any edge). Let $T_n$ be the "triangle" 3-graph of $K_{n+1} - u$. Clearly, since we cut out an open disk, the trace of each vertex in $T_n$ is a path, except possibly for one cycle. Thus, $T_n$ satisfies minimally the first order trace condition (see the basic lemma (2.1)), becoming a candidate for minimal tightness. For $n \leq 10$, we can deduce that $T_n$ is tight. On the other hand, the smallest example that we know of of a nontight $T_n$ is of order $n = 15$ [2]. What happens in between is not clear.

On the other hand, let us point out how examples of minimal surface embeddings of complete graphs follows from our construction of $\mathcal{L}_p$ (see Lemma 3.2), in the case when, in addition to the condition that $2n + 1 = p$ is prime, we also have that 2 generates the multiplicative group $\mathbb{Z}_p^*$. For then, the boundary (i.e., edges in only one triangle) consists of a single cycle that can be coned to a new vertex, giving us a triangular embedding of $K_{n+1}$ into a surface. As far as we know, these are new examples of triangular embeddings of complete graphs, which, for small $n$, coincide with the constructions of Ringel and Youngs (see [7]), but that differ thereafter.

How does the known archive of complete triangular embeddings stand the test of tightness, is also a mystery for general $n$.

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