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TIGHT AND UNTIGHT TRIANGULATIONS OF SURFACES BY COMPLETE GRAPHS

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Abstract. Triangular embeddings of complete graphs into surfaces are studied through the notion of tightness which is a natural combinatorial generalization of connectedness for graphs. By means of a construction which "couples" two such surfaces to produce a new one, the existence of untight complete triangular embeddings is proved and the known archive of tight ones is broadened. In particular, K_{30} admits a tight and an untight triangular embedding into the same surface. Therefore, complete graphs may triangulate the same surface in nonisomorphic ways.

1. Tightness and Triangular Embeddings of Complete Graphs.

Embeddings of the complete graph K_n into surfaces have been intriguing to graph theorists for over a century. It took that time to produce an example of minimal genus for each order n as expected by Heawood, see [4]. Here, we are mainly interested in triangular embeddings of K_n, which we also refer to as *complete triangulated surfaces* or *complete triangular embeddings*. What is essentially known is that with the remarkable exception of K_7 which is not embeddable in the Klein bottle, there exist complete triangular embeddings whenever the "numbers coming from Euler's formula permit". But for a given admissible n, only one such embedding has been constructed, see [4]. The problem of enumerating them remains wide open, see [3].

Our main achievement in classic embedding theory is the coupling construction. Formally, it depends only on the basic definition (1.2), and thus, after the remark following (1.2) the reader may go to Section 4 where this construction is presented. It couples the combinatorial information of two complete triangulated surfaces with boundary of order n (3-chains according to (1.2)) to produce a complete triangulated closed surface of order $2n$

(a 3-cycle according to (1.2)). It yields many triangular embeddings of complete graphs into non-orientable surfaces, which should be new, for the construction does not depend on congruence (mod 6) as Ringel's ones do [5]. Our construction may be seen as an operation among complete triangular embeddings, and it reveals some of the structure, far from being understood, of such triangulations of surfaces.

Our purpose in this work is to study triangular embeddings of K_n into surfaces through the notion of tightness, introduced in [1] for uniform hypergraphs. On one hand, infinite families of tight and untight surfaces are constructed (Theorems 1 and 2). And on the other hand, tightness invariants are used to differentiate triangular embeddings of K_n into the same closed surface –three for $n = 16$, see remark after Theorem 3.6; and two for $n = 30$, see remark after Theorem 2. The smallest n for which there exist non-isomorphic triangular embeddings of K_n into the same closed surface is 9 [6].

In this paper we shall deal with 3-graphs –that is, uniform hypergraphs of rank 3. After the basic general definitions and observations, triangulations of surfaces by complete graphs shall become our particular case of interest.

A 3-*graph* H is defined over a finite set of vertices $V = H^{(0)}$ by a fixed set $H^{(3)}$ of subsets of order 3, called the 3-edges of H. As usual, a 2-*graph* (or simply a *graph*) G consists of a vertex set $G^{(0)}$ together with a fixed 2-edge (or edge) set $G^{(2)}$. (See Section 2 for a comment on our conventions on such superscripts).

(1.1) A 3-graph is called *tight* if every nondegenerate 3-partition of its vertices has a transversal 3-edge; that is, for every surjective function $f\colon H^{(0)} \to \Delta_3 = \{1, 2, 3\}$ there exists $\alpha \in H^{(3)}$ such that $f(\alpha) = \Delta_3$.

Observe that the definition of connectedness for graphs is obtained from (1.1) by replacing 3 with 2. Thus, tightness is a generalization of connectedness.

Let u be a vertex of the 3-graph H. The *trace* $\mathcal{T}(u) := \mathcal{T}_H(u)$ of u in H is the graph defined by

$$\mathcal{T}(u)^{(0)} = H^{(0)} - \{u\} = V - \{u\},$$
$$vw \in \mathcal{T}(u)^{(2)} \iff uvw \in H^{(3)}.$$

A necessary condition for H to be tight is that every such trace must be connected (compare with (2.2)). Otherwise, a 2-partition disconnecting $\mathcal{T}(u)$ together with $\{u\}$ proves H to be *untight*. Observe also that if H is tight, then every pair of vertices lies in some 3-edge because a connected trace has no isolated points. (Of course, we assume here that a 3-graph has order at least 3.)

Now we define the main objects to be considered.

(1.2) A 3-graph H is called a 3-*cycle* (respectively, a 3-*chain*) if, for every vertex $u \in V$, its trace $\mathcal{T}_H(u)$ is a cycle (respectively, a chain –this is, a path).

Remark. 3-cycles correspond precisely to triangular embeddings of complete graphs into closed surfaces. Indeed, the trace of a vertex is a graph defined over all the other vertices, thus, when the 1-skeleton of the associated 2-dimensional simplicial complex is a complete graph, the traces correspond to the usual simplicial links. Similarly, 3-chains correspond to triangular embeddings of complete graphs into surfaces with (not necessarily connected) boundary, in such a way that the boundary contains every vertex and is made up of edges, namely, the edges that join each vertex to the endpoints of its trace.

Examples of 3-cycles of any order $n \equiv 0, 1 \pmod 3$ can be found in the masterful work on Heawood's Conjecture by Ringel et al, [4]. (To see that there are no 3-cycles of order $n \equiv 2 \pmod 3$, count the number of 3-edges from the trace information.) And from these examples, 3-chains of any order $n \equiv 0, 2 \pmod 3$ are obtained by deleting a vertex (clearly, if H is a 3-cycle then $H - v$ is a 3-chain for any vertex v). But it is not known whether these explicit examples are tight or not, except those of small order whose tightness can be established with the help of a computer.

3-chains arose naturally from the study of minimum tight 3-graphs [1]. A tight 3-graph is minimum if it has the minimum *size* (that is, number of 3-edges) among the tight 3-graphs of the same order. If H is a tight 3-graph of order n, then each trace has at least $n-2$ edges because it is connected. Thus, the size of H is at least $\lceil n(n-2)/3 \rceil$. In [1] it was conjectured that minimum tight 3-graphs reach this lower bound, and the first infinite family of examples was presented:

Family 1: *Prime surfaces.* Let $p \geq 7$ be a prime number. Consider the 3-graph $\mathcal{B}_p$, whose vertex set is the multiplicative group $\mathbf{Z}_p^* = \mathbf{Z}_p - \{0\}$, and having a 3-edge $xyz \in \mathcal{B}_p^{(3)}$ whenever $x + y = z$. Clearly, $\mathbf{Z}_p^*$ acts on $\mathcal{B}_p$.

Let $\mathcal{L}_p$ be the quotient 3-graph $\mathcal{L}_p = \mathcal{B}_p/\{-1, 1\}$, whose vertices are the pairs $[\![x]\!] = \{x, -x\}$, $x = 1, 2, \ldots, (p-1)/2$.

$\mathcal{L}_p$ is easily seen to be a 3-chain. Indeed, observe that $\mathcal{T}([\![1]\!])$ is the chain $[\![2]\!], [\![3]\!], \cdots, [\![(p-1)/2]\!]$, and then the multiplicative action of $\mathbf{Z}_p^*$ on $\mathcal{L}_p$ proves that all the other traces are also chains. Furthermore, $\mathcal{L}_p$ is tight (see [1]). $\square$

The main result of this paper is a construction which produces a 3-cycle $\widetilde{H}$ of order $2n$ out of a 3-chain H of order n. By applying it to the 3-chains of Family 1, we shall establish the following existence theorems, which are best stated after a definition.

(1.3) A prime number p is called *connected* if, within $\mathbf{Z}_p^*$, $\{2^k\}_{k\geq 0}$ is transversal to the partition $\{[\![x]\!]\}_{x=1}^{(p-1)/2}$; or equivalently, if the subgroup generated by 2 in $\mathbf{Z}_p^*$ acts transitively on $\mathcal{L}_p$.

Theorem 1. *There exist tight 3-cycles of order $(p+1)/2$ and $p-1$ for all connected primes $p \geq 7$.*

Theorem 2. *There exist untight 3-cycles of order $2^k(p-3)+2$ (and therefore, untight 3-chains of order $2^k(p-3)+1$), for all primes $p \geq 17$ that are not connected, and $k \geq 0$.*

Remark. Note that $30 = (59+1)/2 = 2(17-3)+2$ is the smallest order satisfying both of the theorems. Thus there exist two nonisomorphic triangular embeddings of K_{30} in the same closed surface (which is non-orientable because its Euler Characteristic, -115, is odd).

The proofs of these theorems are completed in Section 4. Section 2 introduces basic elements of tightness theory, relating them to surfaces. Section 3 provides the motivation for the "coupling" construction, which we came across when searching for an untight 3-cycle. It is proved there that the smallest examples of untight 3-cycles have order 16; thus, the least order of Theorem 2 is best possible for 3-cycles. We do not know, however, if there are untight 3-chains of order 14. Finally, in Section 5, we briefly address the question of orientability.

2. Minimal 2-tightness.

We start this section introducing terminology and reviewing basic facts. Although the main ideas follow [1], the generality and emphasis vary. Then, a classic proof of Sperner's Lemma, see [2], is used for a tightness Theorem.

Given a set V, and $k \geq 1$, let $V^{(k)}$ denote the set of all k-sets (sets of order k) of V. A *k-graph* H is a pair $(V, H^{(k)})$, where $H^{(k)}$ is a subset of $V^{(k)}$; V and $H^{(k)}$ are the set of *vertices* of H and the set of *k-edges* of H, respectively. The vertex set of a k-graph H will also be denoted $H^{(0)}$. This convention proves convenient for definitions, and is necessary for 1-graphs (sets with a distinguished subset).

By a *coloring* of a k-graph H, we mean a map from its vertex set $H^{(0)}$ to some color set. And we will emphasize a *k-coloring* if it goes <u>onto</u> a set of order k. A k-edge is said to be *k-colored* or *heterochromatic* if its vertices obtain mutually different colors.

(2.1) A k-graph is called *r-tight* if every k-coloring has at least r k-colored k-edges.

Observe that tightness, as defined in (1.1) and for later usage, corresponds precisely to 1-tightness. And observe also that for graphs ($k = 2$), r-tightness corresponds to the classic notion of r-edge-connectivity.

In Section 1 we introduced the trace of single vertices in 3-graphs. We shall now extend the scope of this term to arbitrary subsets of the vertex set in order to characterize r-tightness.

Let H be a given k-graph on the vertex set V. Let U be any proper subset of V. The *trace* $\mathcal{T}(U) := \mathcal{T}_H(U)$ of U in H is the $(k-1)$-graph with vertex set $\mathcal{T}(U)^{(0)} = V - U$ and edge set:

$$\mathcal{T}(U)^{(k-1)} = \{ \alpha \in (V-U)^{(k-1)} \,:\, \exists\, u \in U \quad \text{for which} \quad \{u\} \cup \alpha \in H^{(k)} \}.$$

We will regard $\mathcal{T}(U)$ as a multi-$(k-1)$-graph, assigning to each $(k-1)$-edge $\alpha \in \mathcal{T}(U)^{(k-1)}$ the *weight* (or multiplicity) $w(\alpha) = \#\{u \in U \,:\, \{u\} \cup \alpha \in H^{(k)} \}$. If in the definition of r-tightness one counts the weights in the obvious manner, we have a general analog of the Basic Lemma [1].

Lemma 2.2. *A k-graph H is r-tight if and only if all of its weighted traces are r-tight.*

Proof. Let a k-coloring of H, $f: V \to \{1, \dots, k\}$, be given. Consider $V_k = f^{-1}(k)$, and observe that the number of k-colored k-edges of H correspond precisely to the sum of weights of $(k-1)$-colored $(k-1)$-edges of $\mathcal{T}(V_k)$ with the obvious restricted coloring. The lemma easily follows from this fact. □

From now on we consider the case $k = 3$.

The case $r = 1$ was studied in [1]. Analogously, for $r = 2$ the Basic Lemma gives us a lower bound on the size of a 2-tight 3-graph H of order n. Indeed, since the minimum 2-tight 2-graphs are cycles, then each vertex of H lies in at least $(n-1)$ 3-edges (corresponding to the 2-edges of its trace). Thus H has at least $\lceil n(n-1)/3 \rceil$ 3-edges.

Conjecture 2.3. For every $n \geq 4$, and $n \equiv 0, 1 \pmod 3$, there exist 2-tight 3-graphs of order n and size $n(n-1)/3$.

This conjecture clearly leads to the study of 3-cycles (recall definition (1.2)) because the one-vertex traces must be minimum 2-tight graphs (i.e., cycles). The problem, according to the Basic Lemma, is that all of the higher order traces of a 3-cycle should be also checked to be 2-tight. Remarkably, their surface structure reduces this "checking" to simple tightness, or connectedness, for higher order traces:

Theorem 2.4. *If a 3-cycle is 1-tight then it is 2-tight.*

Proof. Let $H = (V, H^{(3)})$ be a 3-cycle. Consider the *dual* graph of H: let D $(= D(H))$ be the graph with vertex set $H^{(3)}$ (that is, $D^{(0)} = H^{(3)}$), and with an edge $\alpha\beta$ whenever $\#(\alpha \cap \beta) = 2$. Because H is a 3-cycle, we have that D is a cubic (regular of degree 3) graph, and the function:

$$\begin{aligned} D^{(2)} &\longrightarrow V^{(2)} \\ \alpha\beta &\longmapsto \alpha \cap \beta \end{aligned} \tag{2.5}$$

is a bijection.

Now, suppose $f: V \to \Delta = \{0,1,2\}$ is a given 3-coloring of H. Let D_f be the spanning subgraph of D with an edge $\alpha\beta \in D_f^{(2)} \subset D^{(2)}$ if and only if $f(\alpha \cap \beta) = \{0,1\}$ (any other pair of colors serves as well). Observe that a vertex α of D_f has degree 1 if and only if the 3-edge α is heterochromatic (otherwise it has degree 0 or 2). Since the number of odd degree vertices in a graph is even, then, whenever f yields a heterochromatic 3-edge, it produces at least one more such a 3-edge, and the theorem follows. $\square$

This Theorem is merely an instance of how Sperner's Lemma can be generalized from the disk to more general triangulated surfaces with colorings; the proof given here follows [2]. For our present purposes, it establishes Conjecture 2.3 affirmatively for $n \le 15$ and $n \equiv 0,1 \pmod 3$, (Theorem 3.6), and for $n = p-1$, $(p+1)/2$ with p a connected prime (Theorem 1).

3. The structure of untight 3-cycles.

Let H be a 3-cycle over the vertex set V of order n. Suppose H is untight. Then there exists an *untight* coloring of H, that is, a 3-coloring $f: V \to \Delta = \{0,1,2\}$ without a heterochromatic 3-edge. Such a coloring f, to be fixed hereafter, is equivalent to the partition $\{V_0, V_1, V_2\}$ of V with $V_i = f^{-1}(i)$, which has no transversal 3-edge.

Let $n_i = \# V_i$; so that $\sum_{i=0}^{2} n_i = n$. Clearly we may assume that

$$n_0 \le n_1 \le n_2 \ , \tag{3.1}$$

and we shall call this sequence the *type* of the untight coloring (or partition). With the above notation and assumptions, we now prove some constraints on the type.

Lemma 3.2. $n_0 \ge 3$.

Proof. It suffices to see that all two-vertex traces of the 3-cycle H are connected. Consider a pair of vertices $\{u,v\}$, and observe that $\mathcal{T}(u) - v \subset \mathcal{T}(\{u,v\})$. Since $\mathcal{T}(u) - v$ is a chain (it is a cycle minus a vertex) and it spans $\mathcal{T}(\{u,v\})$ (they have the same vertex sets), then the latter is connected. $\square$

Proposition 3.3.

$$\sum_{i=0}^{2} n_i^2 \ \geq \ \sum_{i<j} n_i n_j + n \quad .$$

Proof. First, we claim that for $i \neq j$, we have:

$$n_i n_j \ = \ \#\{\, \alpha \in H^{(3)} : \ f(\alpha) = \{i, j\}\,\}. \tag{3.3.1}$$

To see this, let D be the dual graph of H, defined in the proof of Theorem 2.4. Let $D_{i,j}$ be the subgraph of D defined by:

$$D_{i,j}^{(0)} = \{\, \alpha \in H^{(3)} : f(\alpha) = \{i, j\}\,\}\,,$$
$$D_{i,j}^{(2)} = \{\, \alpha\beta \in D(H)^{(2)} : f(\alpha \cap \beta) = \{i, j\}\,\}\,.$$

Since f has no heterochromatic 3-edges, $D_{i,j}$ is well defined. Indeed, each pair of vertices colored with i and j appears in exactly two 3-edges, both colored only with i and j. (In particular, $D_{0,1}$ is obtained from the graph D_f, defined in the proof of Theorem 2.4, by deleting all isolated vertices.)

Since $D_{i,j}$ is clearly regular of degree 2, it has the same number of vertices as of edges. And because of the bijection (2.5), this number is $n_i n_j$. Therefore, (3.3.1) holds.

Finally, the 3-edges that appear in (3.3.1) for all choices of $\{i, j\}$ are a subset of the 3-edges in H, so that:

$$\sum_{i<j} n_i n_j \ \leq \ \frac{n(n-1)}{3} \quad .$$

And the proposition is a simple restatement of this inequality. □

Now we define weighted graphs over the vertex sets V_i as follows. Let G_j^i be the weighted subgraph of $\mathcal{T}(V_j)$ generated by V_i. Explicitly, for $vw \in V_i^{(2)}$, G_j^i has weight function:

$$g_j^i(vw) = \#\{u \in V_j \ : \ uvw \in H^{(3)}\}$$

Let $q(G_j^i)$ denote the number of edges of G_j^i, that is, the sum of weights taken over all $vw \in V_i^{(2)}$.

Lemma 3.4. *With the above notation, and* $\{i, j, k\} = \{0, 1, 2\}$*, we have:*

a) $q(G_j^i) \geq n_i$

b) $q(G_j^i) + q(G_i^j) = n_i n_j$

c) $q(G_j^i) + q(G_k^i) \leq n_i(n_i - 1)$

Proof. **(c)** follows because every pair $vw \in V_i^{(2)}$ is in two 3-edges of H, and then $g_j^i(vw) + g_k^i(vw) \leq 2$, (observe that the difference is the number of $\{i\}$-monochromatic 3-edges on which vw lies).

(b) is basically a restatement of equation (3.3.1). Indeed, each α in the right hand side set, counts as an edge in G_j^i if $\#(\alpha \cap V_i) = 2$; otherwise it counts for G_i^j.

To prove **(a)**, we see that the degree of any vertex in G_j^i is at least 2. So let $u \in V_i$. Pick any $v \in V_j$, and let $\alpha\beta$ be the edge dual to uv in $D_{i,j}$ (see previous proof). Recall, that $D_{i,j}$ is a union of cycles, and let C be the component of α. If every $\gamma \in C^{(0)}$ corresponded to a 3-edge with two vertices in V_j, then $\mathcal{T}(u)$ would have a proper cycle component, which can't happen. Thus, there exist $\gamma \in C^{(0)} \subset H^{(3)}$ for which $\#(\gamma \cap V_i) = 2$. The two such γ's closest to the edge $\alpha\beta$ in both directions (which must be different) account for two edges of G_j^i incident to u. □

Proposition 3.5. $n_0 + n_1 \geq 8$.

Proof. In view of (3.1) and (3.2), we must only rule out the cases $n_0 = 3$ with $n_1 = 3$, 4.

Suppose $n_0 = 3$. From **(a)** and **(c)** (of Lemma 3.4) with $i = 0$, we clearly obtain that $q(G_1^0) = 3$. Then, with $j = 1$, **(b)** implies

$$q(G_0^1) = 3(n_1 - 1) \ .$$

Finally, **(a)** and **(c)** with $i = 1$, give

$$n_1 \leq q(G_2^1) \leq n_1(n_1 - 1) - q(G_0^1) = (n_1 - 3)(n_1 - 1) \ ,$$

which is only possible for $n_1 \geq 5$. □

Theorem 3.6. *The smallest order of an untight* 3*-cycle is* 16.

Proof. From (3.2), (3.3) and (3.5) it follows that the smallest possible untight partitions are of type $3, 5, 8$ and $4, 4, 8$, thus yielding the lower bound of 16.

Now we define $S_{3,5,8}$, an untight 3-cycle of type $3, 5, 8$. Let $V_0 = \{a, b, c\}$, $V_1 = \{1, 2, 3, 4, 5\}$ and $V_2 = \{a', b', c', 1', 2', 3', 4', 5'\}$. And then, over $V = \bigcup V_i$, define $S_{3,5,8}$ with 3-edges to be the triangles in the three annuli of Figure 1 (which correspond to the $D_{i,j}$'s) plus one monochromatic 3-edge (123).

It is not hard to convince oneself that $S_{3,5,8}$ is a 3-cycle. One should identify (in pairs) the boundary edges of the three annuli and the triangle (123) of Figure 1, as prescribed by the labelling of the vertices. Then it is a simple matter to verify that we thus obtain a triangular embedding of K_{16} in a closed surface. It has the untight partition V_0, V_1, V_2. □

Another way to check that $S_{3,5,8}$ is indeed a 3-cycle is using Lemma 4.2 below by observing that it is an example of a "coupling" (see the paragraph preceding (4.2)).

Remark. There is also an untight 3-cycle of type $4, 4, 8$. It is $\mathcal{C}_{17,0}$ of Family 2 defined in the next section. It is not isomorphic to $S_{3,5,8}$ because all the traces of three vertices in $\mathcal{C}_{17,0}$ are connected. It is also true that all the four vertex traces of $S_{3,5,8}$ are connected. Hence, though they are both untight, their untight partition types differentiate them. There is yet another triangular embedding of K_{16} in the same non-orientable surface: Ringel's one (see [4] pg . 139). It turns out to be tight, and thus not isomorphic to the previous two. (These facts were proved by exhaustive computer search checking connectedness of traces.)

$S_{3,5,8}$:

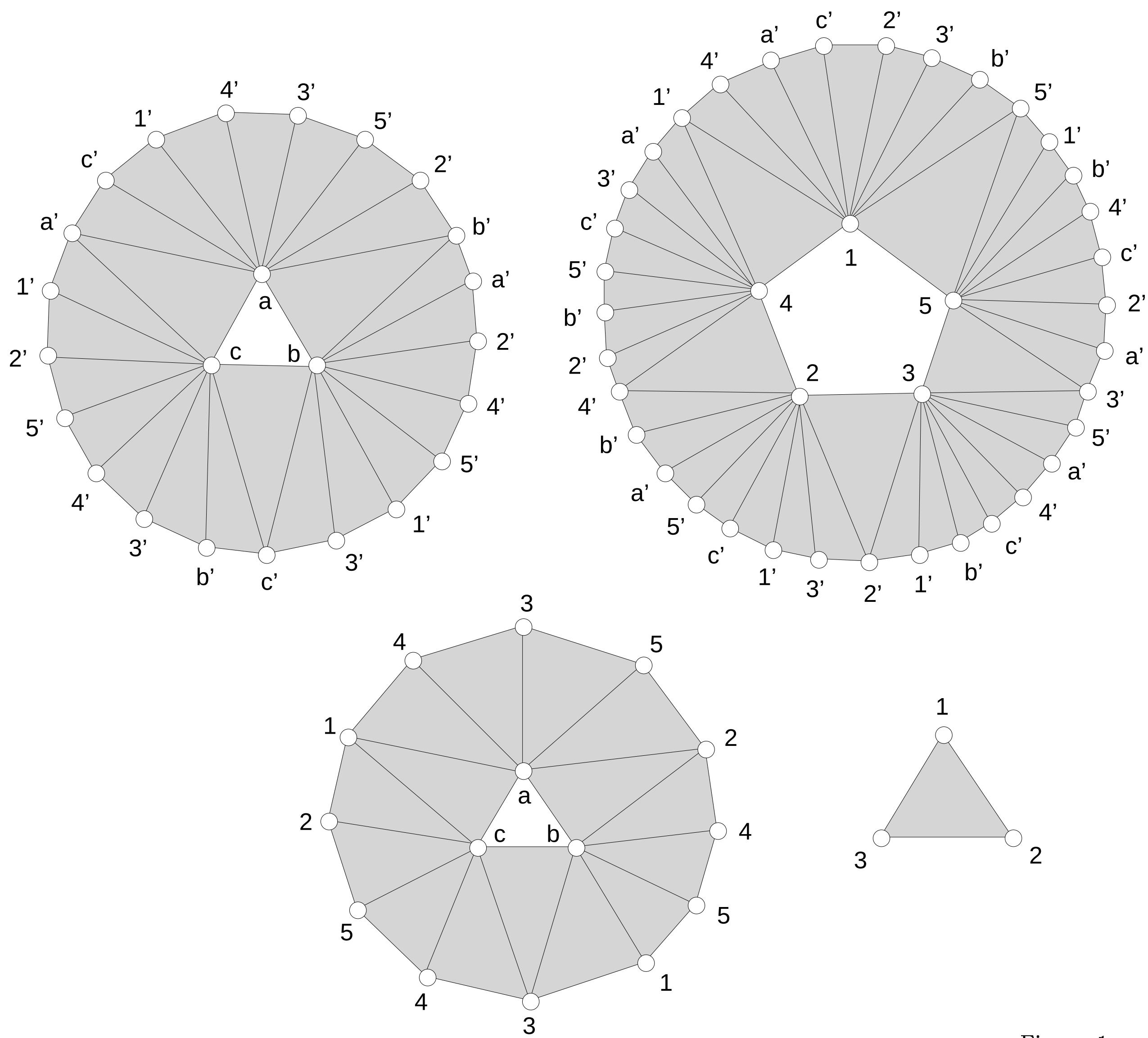

Figure 1.

4. The coupling construction.

Let H be a 3-chain. The *boundary* of H, denoted ∂H, is the graph over the vertex set V ($\partial H^{(0)} = H^{(0)}$), with an edge $uv \in \partial H^{(2)}$ whenever v is an endpoint of the chain $\mathcal{T}_H(u)$ (which is clearly a symmetric relation). By definition, ∂H is regular of degree 2, and thus a disjoint union of cycles.

Let H_0 and H_1 be 3-chains over the vertex sets V_0 and V_1 respectively. Let $\varphi\colon \partial H_1 \to \partial H_0$ be a graph isomorphism, (this implies that both H_0 and H_1 have the same order, n say). Suppose that ∂H_1 is provided with an orientation on each of its cycle components, and let us denote by $\overrightarrow{\partial H}_1$ the corresponding digraph, which is a union of oriented cycles.

Definition 4.1. The *coupling of H_0 and H_1 along φ* is the 3-graph $\widetilde{H} = H_0 \diamond_\varphi H_1$ with vertex set $\widetilde{V} = V_0 \sqcup V_1$, (the disjoint union of the vertex sets), and 3-edges defined by:

- **c1)** $uvw \in H_0^{(3)} \quad \Rightarrow \quad uvw \in \widetilde{H}^{(3)}$.
- **c2)** $xyz \in H_1^{(3)} \quad \Rightarrow \quad \varphi(x)yz \in \widetilde{H}^{(3)}$, $x\varphi(y)z \in \widetilde{H}^{(3)}$ and $xy\varphi(z) \in \widetilde{H}^{(3)}$.
- **c3)** $\overrightarrow{xy} \in \overrightarrow{\partial H}_1^{(2)} \quad \Rightarrow \quad \varphi(x)\varphi(y)y \in \widetilde{H}^{(3)}$ and $xy\varphi(y) \in \widetilde{H}^{(3)}$.

For a 3-chain H, the *coupling* of H is $H \diamond H = H \diamond_{\mathrm{id}} H$ where id is the identity map of ∂H, which is supposed to have a preferred orientation.

As examples, observe that the coupling of a triangle ($n = 3$), yields the 3-cycle corresponding to a triangular embedding of K_6 in the projective plane. Or consider $S_{3,5,8}$ defined in Theorem 3.6. Observe that the subhypergraph generated by $\{a, b, c, 1, 2, 3, 4, 5\}$, that is, the 3-edges of the bottom of Figure 1, is a 3-chain, H say. Its boundary consists of two cycles, which oriented (a, b, c) and $(1, 5, 3, 2, 4)$, yield $S_{3,5,8} = H \diamond H$. The primes on the other eight vertices of $S_{3,5,8}$ obey to the necessity of taking two disjoint copies of the vertices to perform the coupling of a 3-chain with itself.

As a matter of notation, vertices in H_0 are denoted by u, v and w; while x, y and z are used for vertices in H_1. Also, we abbreviate $\mathcal{T}_{H_i}$ ($i = 0, 1$) and $\mathcal{T}_{\widetilde{H}}$ by $\mathcal{T}_i$ and $\widetilde{\mathcal{T}}$ respectively.

Lemma 4.2. $\widetilde{H} = H_0 \diamond_\varphi H_1$ *is a 3-cycle.*

Proof. We must show that all one-vertex traces of $\widetilde{H}$ are cycles, which gives us two cases.

Case 1 : $u \in V_0$. From **(c1)**, we have a canonical inclusion $H_0 \hookrightarrow \widetilde{H}$. And thus, we have a subchain $\mathcal{T}_0(u) \hookrightarrow \widetilde{\mathcal{T}}(u)$.

Let $x = \varphi^{-1}(u)$. Then, **(c2)** gives us another subchain $\mathcal{T}_1(x) \hookrightarrow \widetilde{\mathcal{T}}(u)$. Thus far, we have accounted for all the vertices in $\widetilde{\mathcal{T}}(u)$ except x. The orientation of $\overrightarrow{\partial H}_1$ gives us operators $^+$ and $^-$ on V_1 (and, through φ, on V_0), defined by $\overrightarrow{x^-x}, \overrightarrow{xx^+} \in \overrightarrow{\partial H}_1^{(2)}$. Observe that x^- and x^+ (u^- and u^+) are the endpoints of $\mathcal{T}_1(x)$ ($\mathcal{T}_0(u)$). Then, the three possible appearances of u in the right hand side of **(c3)** (namely, u^-ux, x^-xu and uu^+x^+) give us the edges that complete the cycle $\widetilde{\mathcal{T}}(u)$; see Figure 2.

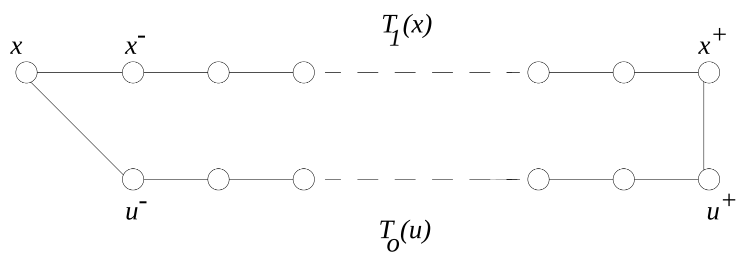

Figure 2.

Case 2 : $x \in V_1$. As before, let $u = \varphi(x)$. Let $\rho\colon \widetilde{V} \to V_1$ be the canonical projection; that is, $\rho(y) = y$ for $y \in V_1$, and $\rho(v) = \varphi^{-1}(v)$ for $v \in V_0$. Observe that $\rho \,|\, \widetilde{V} - \{x, u\}$ gives a 2-fold graph covering $\rho\colon \widetilde{\mathcal{T}}(x) - u \to \mathcal{T}_1(x)$, which degenerates only over x^+. Indeed, each edge $yz \in \mathcal{T}_1(x)$ is covered twice, by $\varphi(y)z$ and $y\varphi(z)$, because of **(c2)**. When we add u and the two edges it carries because of **(c3)**, we obtain the cycle $\widetilde{\mathcal{T}}(x)$; see Figure 3.

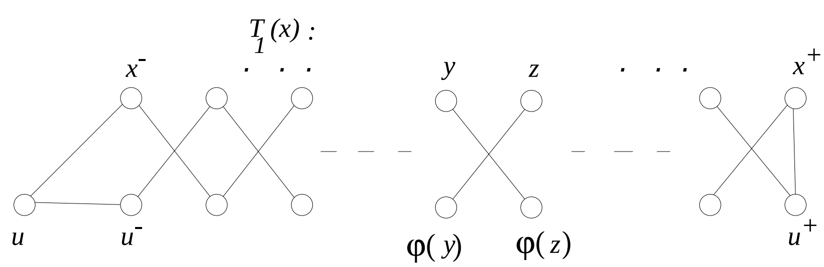

Figure 3.

Since no other edges may appear in $\widetilde{\mathcal{T}}(x)$, this concludes the proof. □

Family 2: *Iterated prime surfaces.* Let p be any prime number. Let $\mathcal{C}_{p,0} = \mathcal{L}_{p,-1} \diamond \mathcal{L}_{p,-1}$, where $\mathcal{L}_{p,-1} = \mathcal{L}_p$ is the 3-chain of Family 1. By the preceding lemma, $\mathcal{C}_{p,0}$ is a 3-cycle of order $(p-1)$. Now we will iterate $k \geq 0$. For this, take the 3-chain

$$\mathcal{L}_{p,k} = \mathcal{C}_{p,k} - x \ ,$$

where x is a vertex of $\mathcal{C}_{p,k}$, and then take the 3-cycle

$$\mathcal{C}_{p,k+1} = \mathcal{L}_{p,k} \diamond \mathcal{L}_{p,k} \ .$$

□

In principle, the choice of a vertex to be deleted and of an orientation on the boundary to perform a coupling, affect the isomorphism class of the iterated prime surfaces. That is, different choices may lead to non-isomorphic hypergraphs. For our present purposes we do not need to make these choices precise. Thus, $\mathcal{L}_{p,k}$ and $\mathcal{C}_{p,k}$ are defined only as elements of non-void classes, because the choices can obviously be made for all $k \geq 0$.

Observe also that the family of 3-cycles $\{\mathcal{C}_{p,k}\}$ may be extended to $k = -1$, when p is connected. Indeed, for arbitrary p the components of $\partial\mathcal{L}_p$ correspond to the orbits of the group $\langle 2 \rangle \subset \mathbf{Z}_p^*$ acting on $\mathcal{L}_p^{(0)} = \mathbf{Z}_p^*/\{1,-1\}$ (the operator $^+$ in the proof of Lemma 4.2 may be taken to be multiplication by 2 in this case). So that when $\partial\mathcal{L}_p$ is connected (that is, when p is connected, recall (1.3)) we can annihilate the boundary by adding a new vertex, together with the 3-edges defined by that vertex and the original boundary edges; we thus obtain a 3-cycle $\mathcal{C}_{p,-1}$ to initiate the iteration process. However, in the case $\partial\mathcal{L}_p$ is not connected, there exists an infinite family of untigtht 3-cycles.

Lemma 4.3. *If $\partial H_0 \cong_\varphi \partial H_1$ is not connected then $\widetilde{H} = H_0 \diamond_\varphi H_1$ is not tight.*

Proof. Observe that $\widetilde{\mathcal{T}}(V_1) = \partial H_0$. Indeed, the inclusion $H_0 \hookrightarrow \widetilde{H}$ gives us equality at the vertex level. And the only 3-edges of $\widetilde{H}$ with exactly one vertex in V_1 arise from the first part of **(c3)**, giving ∂H_0 at the trace level. Now, the Basic Lemma 2.2, asserts that $\widetilde{H}$ is untight if any of its traces is untight. □

Since $\mathcal{L}_{17}$ has two boundary components, $([\![1]\!],[\![2]\!],[\![4]\!],[\![8]\!])$ and $([\![3]\!],[\![6]\!],[\![5]\!],[\![7]\!])$, then $\mathcal{C}_{17,0} = \mathcal{L}_{17} \diamond \mathcal{L}_{17}$ is untight, having the minimum possible order (by Theorem 3.6). Its untight partition is of type $4, 4, 8$ as mentioned in Section 3.

Now we proceed to establish sufficient conditions for a coupling to be a tight 3-cycle.

Theorem 4.4. *Let H_0 and H_1 be tight 3-chains with tight boundaries. Then, for any isomorphism $\varphi\colon \partial H_1 \to \partial H_0$ and any orientation of ∂H_1, $\widetilde{H} = H_0 \diamond_\varphi H_1$ is tight.*

Proof. Consider an arbitrary 3-coloring of H with colors **1**, **2** and **3**. In case all three colors occur in V_0, there certainly can be found a heterochromatic 3-edge in the tight 3-chain $H_0 \hookrightarrow \widetilde{H}$. Assume one of the colors, say **3**, is omited from V_0. Then a heterochromatic 3-edge still can be found provided $\widetilde{\mathcal{T}}(X)$ is connected, where X is the set of the vertices of V_1 colored with **3**. We shall now prove that $\widetilde{\mathcal{T}}(X)$ is connected for any subset X of V_1, by exposing a connected spanning subgraph G of $\widetilde{\mathcal{T}}(X)$. Since $\widetilde{\mathcal{T}}(V_1) = \partial H_0$ is connected by hypothesis, we shall assume $X \neq V_1$.

Let $U = \varphi(X)$, and consider the induced subgraph of $\widetilde{\mathcal{T}}(X)$ generated by U; denote it $G_0 = \langle U : \widetilde{\mathcal{T}}(X) \rangle$. Since the only edges it has, come from the first part of **(c3)**, we obtain that $G_0 = \langle U : \partial H_0 \rangle$. Thus, being an induced proper subgraph of an oriented cycle, G_0 is a union of oriented chains (some of them possibly of length 0).

On the other hand, consider the graph G_1 with vertex set $\widetilde{V} - X - U$, and with two edges $\varphi(y)z$ and $y\varphi(z)$ for every edge $yz \in \mathcal{T}_1(X)^{(2)}$. Clearly G_1 is a subgraph of $\widetilde{\mathcal{T}}(X)$.

From the construction, G_1 comes equipped with a projection

$$\rho\colon G_1 \to \mathcal{T}_1(X) \ ,$$

giving it the structure of an alternating 2-fold covering. Because H_1 is tight, $\mathcal{T}_1(X)$ is connected; thus, we may conclude that G_1 is connected unless $\mathcal{T}_1(X)$ is bipartite. But in this case, G_1 has exactly two components transversal to each fiber.

Finally, let G be the union of G_0 and G_1, plus the edges that appear inside $\widetilde{\mathcal{T}}(X)$ because of **(c3)**. G is a spanning subgraph of $\widetilde{\mathcal{T}}(X)$, so that we are left to prove it is connected.

Let $u = \varphi(x)$ be a vertex of G_0 which is the initial end-point of one of its components. Using the notation of Lemma 4.2, let u^- precede u in $\overrightarrow{\partial H}$; observe that $u^- \in G_1$. Since $x = \varphi^{-1}$ [illegible] gives us th[illegible] edges $u^- u$ and $x^- u$ i[illegible].

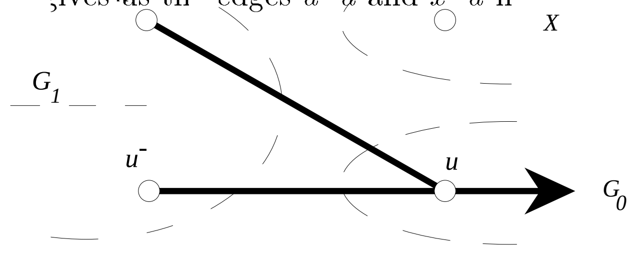

Figure 4.

Thus, the two possible components of G_1 have been connected through u, and the chosen chain component of G_0 has been attached to G_1 by its initial end-point. Since this holds for any component of G_0, it proves that G is connected. The proof is complete. □

Corollary 4.5. *Let H be a 3-chain. Then, the following are equivalent:*

- **a)** *H and ∂H are tight.*
- **b)** *$H \diamond H$ is tight.*

Proof. $\underline{a \Rightarrow b}$: Theorem 4.4.

$\underline{b \Rightarrow a}$: By Lemma 4.3, ∂H is tight. And H is tight because we clearly have a quotient map $H \diamond H \to H$, (see (2.3) of [1]). □

With this result and the examples of Family 2, we may conclude our work from the Introduction.

Proof of Theorem 1. Since $\mathcal{L}_p = \mathcal{L}_{p,-1}$ is tight, and so is its boundary for connected primes, we have that $\mathcal{C}_{p,0}$ is tight. And $\mathcal{C}_{p,-1}$ is also tight because it was obtained by attaching a vertex with a tight trace to a tight 3-graph. □

Proof of Theorem 2. For nonconnected primes, Lemma 4.3 implies that $\mathcal{C}_{p,0}$ is not tight. If we delete any vertex from an untight 3-cycle we obtain an untight 3-chain (see previous proof). From this fact and Corollary 4.5 we conclude inductively that $\mathcal{L}_{p,k}$ and $\mathcal{C}_{p,k}$, for $k \geq 0$, are untight. □

Finally, observe that if we could add to Corollary 4.5 a third item, **c)** *$(H \diamond H) - x$ is tight for some $x \in (H \diamond H)^{(0)}$* , then an inductive proof similar to that of Theorem 2 could be used to strengthen Theorem 1. This would require a careful choice of the vertices to be deleted in the iteration process.

Conjecture 4.6. Given a connected prime p, there exists $k(p) \geq 0$, for which we can make $\mathcal{L}_{p,k}$ be tight for all $k(p) \geq k \geq 0$, by judiciously choosing the vertices x to be deleted while constructing the sequence $\{\mathcal{L}_{p,k}\}_{k\geq 0}$.

In particular, we suspect that $\mathcal{L}_{p,0} = (\mathcal{L}_p \diamond \mathcal{L}_p) - x$ is tight when x is chosen from the second summand. It would be interesting to determine for which connected primes is this conjecture true, and whether $k(p)$ is bounded.

5. On orientability.

Regardless of the summands, the coupling $\widetilde{H} = H_0 \diamond_\varphi H_1$ is non-orientable. Indeed, for any 3-edge $xyz \in H_1^{(3)}$, the simplicial path x, y, z, x reverses orientation in $\widetilde{H}$. To see this, observe from the trace of x (Figure 3), that if $xy\varphi(z) \in \widetilde{H}^{(3)}$ is given the orientation $(x, y, \varphi(z))$, say, then $x\varphi(y)z$ gets the orientation $(x, z, \varphi(y))$ if x is to be locally oriented (and thus, $\widetilde{\mathcal{T}}(x)$). Then, the same argument based on y and z would give opposite orientations to $\varphi(x)yz$. This is illustrated in Figure 5 which presents the three 3-edges of $\widetilde{H}$ arising from $xyz \in H_1^{(3)}$. It remains as an open problem whether there exists an orientable untight 3-cycle.

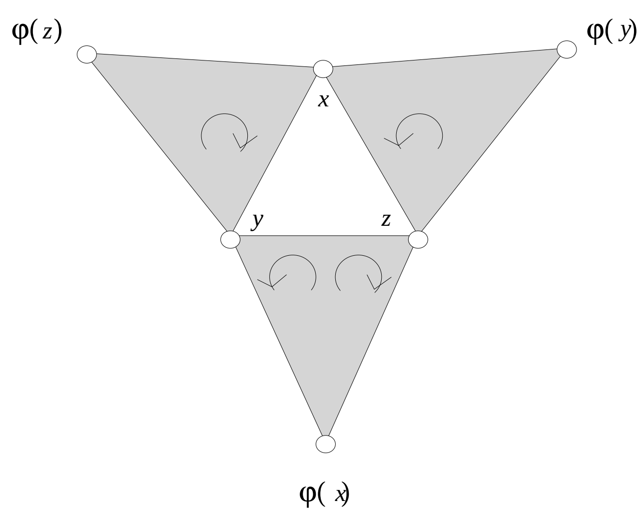

Figure 5.

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