A Quick Proof of Höbinger-Burton-Larman's Theorem

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Abstract

Through the notion of projective center of symmetry of a convex body we will give a quick proof and clarify the ideas surrounding Höbinger's problem, originally proved by Burton and Larman in [1].

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Before stating the Theorem we require a definition.

A *slab* in E^n, in the direction $u \in S^{n-1}$, is a set of the form $\{x \in E^n \mid \delta \leq \langle x, u \rangle \leq \alpha\}$. The slab is said to be degenerate if $\delta = \alpha$ and in this case it is just a hyperplane parallel to the subspace $H_u = \{x \in E^n \mid \langle x, u \rangle = 0\}$.

Höbinger-Burton-Larman's Theorem (HBL). Let $K \subset E^n$, $n \geq 3$, be a convex body containing the origin and let $S_1, ..., S_{n-1}$ be slabs, at least one of which is not degenerate, in the linearly independent directions $u_1, ..., u_{n-1}$, respectively, such that $K \cap S_i$ is empty. Suppose that for each point $P \in S_i$, the projection of K from P into H_{u_i} is centrally symmetric, $1 \leq i \leq n-1$. Then K is an ellipsoid.

Let us consider *projective space* P^n as *euclidean space* E^n plus points at infinity in such a way that $E^n \subset P^n$. Let $K \subset E^n$ be a convex body with nonempty interior. A point P (a hyperplane H) is called a *projective center* of K (a *harmonic hyperplane of K relative to P*) if there is a projective isomorphism $f : P^n \to P^n$ such that $f(K) \subset E^n$ is a centrally symmetric convex body with centre $f(P)$ and $f(H)$ is the hyperplane at infinity.

The following theorem justifies the notion "harmonic hyperplane".

Theorem 1 *A hyperplane H is a harmonic hyperplane of K relative to a projective center P if and only if for every line L through P, $(A, B; P, T)$ is a harmonic set, where $L \cap bdK = \{A, B\}$ and $L \cap H = T$. That is, if and only if* $\frac{AP}{PB} = -\frac{AT}{TB}$.

Proof If K is *a priori* centrally symmetric with center P and hyperplane at infinity H, then $\frac{AP}{PB} = -\frac{AT}{TB} = 1$. The result holds by the fact that harmonic sets are preserved by projective isomorphisms. □

From the above theorem it is straightforward to prove that **the set of all projective centers of a convex body is a closed subset of its interior**.

Next, we expose the significance of harmonic hyperplanes for the proof of HBL Theorem.

Theorem 2 *Let H_0 be a hyperplane such that $H_0 \cap K = \emptyset$ and let H be any other hyperplane parallel to H_0. Then, H_0 is a harmonic hyperplane of K if and only if for every point $P \in H_0$, the projection of K from P into H is centrally symmetric.*

Proof The theorem is the projective version of the fact that a convex body is centrally symmetric if and only if all projections, from a point at infinity into a hyperplane, are centrally symmetric. □

For an ellipsoid, every interior point is a projective center. Moreover, this property characterizes ellipsoids. In fact, we shall prove next the following stronger result.

Theorem 3 *Let K be a convex body and let H_o be a hyperplane that intersects the interior of K. If all points of $H_o \cap intK$ are projective centers of K, then K is an ellipsoid.*

Proof If $P \in \mathrm{int}K$ and $A \in \mathrm{bd}K$, then we say that $B \in \mathrm{bd}K$ is the P-*antipode* of A if $A \neq B$ and A, B and P are collinear. Let us first prove that K is smooth and strictly convex. Let $Q_0 \in \mathrm{bd}K$ be a point which is not a corner point of K. Let now $Q \in \mathrm{bd}K$ be such that the open interval (Q, Q_O) contains a projective center P of K. By means of a projective isomorphism f we obtain a centrally symmetric convex body $f(K)$ with center $f(P)$. Since $f(Q_0)$ is not a corner point of $f(K)$ and $f(Q)$ is $f(P)$-antipode of $f(Q_0)$, then $f(Q)$ is not a corner point of $f(K)$ and hence Q is not a corner point of K . Consequently, since all points of $H_o \cap \mathrm{int}K$ are projective centers of K, then K is smooth. Similarly, K is strictly convex.

Let $P_0 \in \mathrm{int}K \cap H_0$ and assume that K is centrally symmetric with center P_0. Let H be a 2-plane through P_o. We shall prove that $\mathrm{bd}(H \cap K)$ is a shadow boundary of K. If this is so, due to the fact that K is centrally symmetric, then every shadow boundary of K is planar, by a theorem due to Blaschke (see Theorem 16.14 of [2]) every 3-dimensional section of K is an ellipsoid and therefore K is an ellipsoid.

In order to prove that $\mathrm{bd}(H \cap K)$ is a shadow boundary of K, let us consider a diametral chord $[A, B]$ of K through P_0 contained in $H \cap H_0$ and hence with the property that every point of (A, B) is a projective center of K. Let Γ be a $(n-3)$-plane at infinity contained in the intersection of the two parallel supporting hyperplanes of K at A and B and let Σ_Γ be the shadow boundary of K with respect to Γ. Since Σ_Γ is the set of all points of $\mathrm{bd}K$ which are in a supporting hyperplane of K through the $(n-3)$-plane Γ, we have that $A,\ B \in \Sigma_\Gamma$. We shall prove that the curve Σ_Γ is planar by proving that for every point $X \in \Sigma_\Gamma$ and every point $P \in (A, B)$, the P-antipode of X is also in Σ_Γ. Let f be a projective isomorphism sending K into a centrally symmetric convex body $f(K)$ with center $f(P)$. Suppose for a moment that $f(\Sigma_\Gamma)$ is a shadow boundary of $f(K)$. Then the $f(P)$-antipode of $f(X)$ in $f(K)$ is also in $f(\Sigma_\Gamma)$, because $f(K)$ is centrally symmetric. Hence, the P-antipode of X in K is in Σ_Γ and consequently, the shadow boundary Σ_Γ is planar. This implies that $\mathrm{bd}(H \cap K) = \Sigma_\Gamma$ for some Γ, because $[A, B] \subset H$.

It just remains to prove that $f(\Sigma_\Gamma)$ is a shadow boundary of $f(K)$. For that purpose observe that $f(\Sigma_\Gamma)$ consists of all points of $\mathrm{bd} f(K)$ which are in a supporting hyperplane of $f(K)$ through the $(n-3)$-plane $f(\Gamma)$. Since $f(K)$ is centrally symmetric with center $f(P)$ and $f(A)$ is $f(P)$-antipode of $f(B)$, then the supporting hyperplanes of $f(K)$ at $f(A)$ and $f(B)$ intersect at infinity and hence $f(\Gamma)$ is at infinity. This implies that $f(\Sigma_\Gamma)$ is a shadow boundary of $f(K)$. With this we conclude the proof of Theorem 3. $\square$

Let G be the the open interval $(-1, 1)$ considered as a topological space. Let us define a sum in G by $a \oplus b = (a + b)/(1 + ab)$, for any $a,\ b \in (-1, 1)$. With this sum G is a welldefined topological abelian group. Moreover the total order of real numbers is compatible with the sum. That is, $a < b$ iff $a \oplus c < b \oplus c$ for every a , b and c in G. The following lemma holds in particular for the group G.

Lemma 1 *Let H be a closed subgroup of a totally ordered topological abelian group. Then either H is cyclic (and therefore discrete) or H is the whole group.*

Proof Suppose H is not cyclic. We start proving that 0 is a limit point of H. For that purpose let β be the least positive element of H and let α be an element not generated by β. Let n be the integer such that $n\beta < \alpha < (n+1)\beta$. Adding $-n\beta$ we obtain $0 < \alpha - n\beta < \beta$, which is a contradiction. By symmetry, 0 is a limit point of H from both sides and by homogeneity of H, every point of H is a limit point from both sides, but this is impossible unless H is the whole group. $\square$

The group G naturally acts on P_1 by the rule $a \oplus x = (a+x)/(1+ax)$, for any $a \in G$ and every $x \in P_1$.The map $f(x) = a \oplus x$ is a projective isomorphism of

P_1. In fact, G is **the group of all projective isomorphisms of P_1 which preserve -1, 1 and the open interval $(-1,1)$.**

Theorem 4 *Let K be a convex body, I a chord of K and T the set of all projective centers of K in I. If T is not empty, then there is a projective isomorphism $f : P^n \to P^n$ sending the interior points of I to $G \subset E^n$ and such that $f(T)$ is either a cyclic subgroup of G or all of G.*

Proof Since T is not empty we assume after a suitable projective isomorphism that K is centrally symmetric with center $0 \in I = [-1,1] \subset P_1 \subset P_n$. By the symmetry of T, we have $a \in T \Rightarrow -a \in T$. Let a and b two distinct points in T. There is a projective isomorphism f which preserves -1, 1 and $(-1,1)$ such that $f(a) = 0$ and $f(K)$ is centrally symmetric. By the above remark, the map f restricted to I has the form $(x-a)/(1-xa)$ and therefore $f(b) = b \ominus a$ is a projective center in $f(T)$. By symmetry $a \ominus b$ is also a projective center in $f(T)$. So $f^{-1}(a \ominus b) = a \ominus b \ominus b \in T$.

Let us denote $2\mathbf{N}\alpha = \{\alpha \oplus \cdots \oplus \alpha$ an even number of times $\}$ and $2\mathbf{Z}\alpha = 2\mathbf{N}\alpha \cup -2\mathbf{N}\alpha$. It is not difficult to show, using the above properties of T, (i) $a \in T \Rightarrow \langle a \rangle \subset T$, and (ii) $a, b \in T \Rightarrow 2\mathbf{Z}a \oplus 2\mathbf{Z}b \subset T$.

By (i), if T is contained in a cyclic subgroup, then it is a cyclic subgroup itself. If not, then it contains two points none of which generates the other one. So, by (ii), it contains a non-cyclic subgroup and therefore using Lemma 1 we obtain that $T = G$. □

The Proof of HBL Theorem

Let K be a convex body which contains the origin and let $S_1 = \{x \in E^n \mid \delta \leq \langle x, u_1 \rangle \leq \alpha\}$ be a nondegenerate slab such that $K \cap S_1$ is empty and for each point $P \in S_1$, the projection of K into H_{u_1} is centrally symmetric. Then, by Theorem 2, the hyperplanes $\{x \in E^n \mid \langle x, u_1 \rangle = \beta\}$, $\delta \leq \beta \leq \alpha$, are harmonic hyperplanes of K whose corresponding projective centers are the points of a closed interval that lies in a chord I of K. Consequently, by Theorem 4, all interior points of the chord I are also projective centers of K. Observe that the harmonic hyperplanes of K relative to the projective centers of I are precisely the hyperplanes of E^n orthogonal to u_1 which do not intersect K.

Let H_2 be a hyperplane contained in the slab S_2. Then, by Theorem 2, H_2 is an harmonic hyperplane of K whose corresponding projective center P_2 is a point of int$K$ which is not in I. Let I^2 be the set of all interior points of K which lie in the plane generated by the chord I and the point P_2. We shall prove that every point of I^2 is a projective center of K.

For every open chord J that passes through P_2 and cuts I there is a projective isomorphism f such that $f(K)$ is centrally symmetric with center $0 = f(J \cap I)$. Let $P_2^J = f^{-1}(-f(P_2))$. Then, P_2^J is a projective center of K. Moreover, we may choose the projective isomorphism f in such a way that P_2^J varies continuously while J varies in the set of all chords of K through P_2. In this way, it is possible to construct a curve τ in I^2 such that $\tau \cap I$ consists of the extreme points of τ and I and such that all its interior points are projective centers. Similarly, using instead of P_2 a projective center of τ, it is possible to construct a curve ρ in I^2 such that $\tau \cap I \cap \rho$ consists of the extreme points of ρ, τ and I and such that all its interior points are projective centers.

Let $Q \in I^2$ and let $\epsilon > 0$. From the set of all open chords of K in I^2 that cross the ϵ-neighborhood of Q we may choose an open chord $\mathcal{J}$ with the property that for every projective isomorphism f with $f(\mathcal{J}) = G$ and $f(\mathcal{J} \cap I) = 0$, the points $f(\mathcal{J} \cap \tau)$ and $f(\mathcal{J} \cap \rho)$ are not contained in a cyclic subgroup of G. By Theorem 4, every point of $\mathcal{J}$ is a projective center of K, which implies that the set of projective centers of K is dense in I^2. Since the set of projective centers of K is a closed subset of the interior of K, every point of I^2 is a projective center of K.

Proceeding in this way with the vectors $u_3, ..., u_{n-1}$, we construct a hyperplane H_0 that intersects the interior of K with the property that every point of int$K \cap H_0$ is a projective center of K. Hence, by Theorem 3, we have that K is an ellipsoid. This concludes the proof of HBL Theorem.

References

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