Tightness problems in the plane

B. Ábrego , J.L. Arocha , S. Fernández-Merchant
and
V. Neumann-Lara

November 17, 1997(revised)

Abstract

A 3-uniform hypergraph is called tight if for any 3-coloring of its vertex set a heterochromatic edge can be found. In this paper we study tightness of 3-graphs with vertex set $\mathbb{R}^2$ and edge sets arising from simple geometrical considerations. Basically we show that sets of triangles with "fat shadows" are tight and also that some interesting sets of triangles with "thin shadows" are tight.

1 Introduction

A *k-graph* is a pair $G = (V, E)$ of its vertex set V and its edge set E. Edges are by definition subsets of V with cardinality k. A k-graph G is called *tight* whenever for any map f from the vertex set onto a set of cardinality k (the colors) there is an edge e of G such that $|f(e)| = k$ (e is *heterochromatic*). This notion was introduced in [1] as a generalization of connectedness of graphs (graphs are 2-graphs and they are tight if and only if they are connected).

In [1] and [2] an important question for finite 3-graphs is studied , namely how "small" can a tight 3-graph be. In [3] some general results about tightness of infinite k-graphs are obtained. However, this paper is the first attempt to study a concrete class of infinite k-graphs from the point of view of their classification into tight and untight k-graphs.

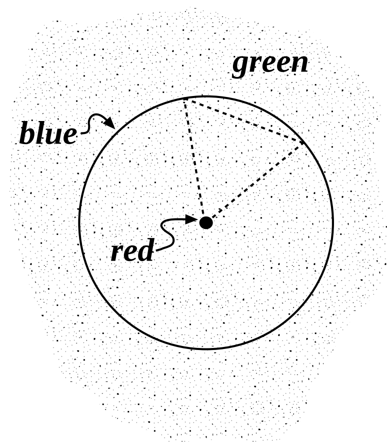
Actually, there is another motivation for this paper. When tightness for a k-graph has to be shown, one must prove that for any "appropriate" coloring there is a heterochromatic edge. On the other hand, it is said that a hypergraph is Ramsey whenever there is a monochromatic edge for any "appropriate" coloring . So, the 3-graph whose vertices are the edges of K_6 and whose edges are the triangles of K_6, is well known to be Ramsey for 2-colorings. Therefore Ramsey properties of hypergraphs are, in some sense, opposite to their tightness properties. An interesting branch of Ramsey Theory initiated by Erdős et. al. [4,5,6] is Euclidean Ramsey Theory (see also [7,8]) which deals with Ramsey properties of hypergraphs arising from geometrical considerations in the n-dimensional Euclidean space. From this point of view our results are included in a branch that could be called "Euclidean Antiramsey Theory".

Here, we study tightness of sets of triangles (triples of non collinear points) in the Euclidean plane $\mathbb{R}^2$. From now on T will denote a set of triangles and we will say that T is tight when the 3-graph $(\mathbb{R}^2, \mathrm{T})$ is tight.

Let T be a set of triangles and let AB be a segment. The set $Sh(AB)$ of all points C in $\mathbb{R}^2$ such that ABC is a triangle in T is called the *shadow* of AB. A subset α of $\mathbb{R}^2$ is a *shadow* of T if there is a segment AB such that $\alpha = Sh(AB)$. It will be always clear which is the set of triangles T.

2 Almost tight sets of triangles

Let T be the set of all equilateral triangles in $\mathbb{R}^2$. By coloring a single point red, coloring blue a circle with center in the red point, and coloring green all other points in $\mathbb{R}^2$ (see figure 1), we obtain a coloring which shows that T is not tight. However, this coloring satisfies a weaker interesting property: there are trichromatic triangles as near as required to an equilateral triangle.



For a fixed coloring of the plane, a triangle ABC is said to be *almost trichromatic* if for every $\varepsilon > 0$ there exist a trichromatic triangle t such that each of the balls with radius ε and centers in A, B and C contains some vertex of t. A set of triangles is said to be *almost tight* if for any coloring of the plane it contains an almost trichromatic triangle.

In the Theorem below we characterize almost tight sets of triangles according to their shadows. Moreover, it turns out that this characterization is useful to prove some criteria for shadows for tightness in the next section.

Theorem 1 *A set of triangles is almost tight if and only if none of its shadows are empty.*

Proof Suppose that no shadow of the set of triangles T is empty. Let us consider some green, blue, red-coloring of the plane. By a point of type blue-green (blue-red, green-red) we mean a point which is at the same time a limit point of blue (blue, green) points and also a limit point of green (red, red) points. Let P, Q, R be three non collinear points with different colors (it is easy to see that such points exist). Thus on the union of the segments PQ, QR and RP there exist points of at least two different types. So, we may assume that A and B are two points of different types (say A is blue-red and B is green-red). Let C be a blue point (the other cases are analogous) on the shadow of AB. Then for any sufficiently small $\varepsilon > 0$ there exist a red point $A_\varepsilon \in Ball_\varepsilon(A)$ and a green point $B_\varepsilon \in Ball_\varepsilon(B)$, therefore the triangle $A_\varepsilon B_\varepsilon C$ is trichromatic and hereby the " if part " of the Theorem is proved.

On the other hand, let AB be a segment such that $Sh(AB)$ is empty. Let us color A with green, B with red and the rest of the plane with blue. Suppose that $t \in T$ is an almost trichromatic triangle for this coloring. We have that $\{A, B\}$ is not contained in t. So, a point (say A) in $\{A, B\}$ is not in t and it is easy to see that for sufficiently small ε , A is not in the ε-neighborhood of any vertex of t. This is a contradiction. $\square$

The elegant formulation of the preceding Theorem is not suitable for its use in the next section. Actually, we proved the following stronger fact in the "if part".

Theorem 2 *Suppose that for a set of triangles none of its shadows are empty. Then there exist points A and B such that for every C in the shadow of AB there are two functions $\mathbb{R}^+ \ni \varepsilon \mapsto \mathrm{A}_\varepsilon \in \mathbb{R}^2$ and $\mathbb{R}^+ \ni \varepsilon \mapsto \mathrm{B}_\varepsilon \in \mathbb{R}^2$ such that the distances between A_ε and B_ε to A and B respectively are smaller than ε and the triangle $A_\varepsilon B_\varepsilon C$ is trichromatic. Moreover, it is always possible to find those functions in such way that their images are monochromatic sets.*

3 Shadow's criteria

Unfortunately, there is a big difference between almost tight sets and tight sets. Namely, we were not able to find the characterization of the latter by properties of their shadows. However, in this section we show that if shadows are sufficiently "thin" (resp. "fat") then the set of triangles is untight (resp. tight). The theorems of this section remain valid in a more general setting of topological spaces but we are not interested now in such generality.

By a shadow-closed set we mean a proper subset S of the plane with at least two points such that the shadow of every pair of points in S is contained in S.

Theorem 3 *If a set of triangles has a shadow-closed set, then it is untight.*

Proof Let S be a shadow-closed set. Since S is a proper subset of the plane we may color it with blue and green and the rest of the plane with red. Thus every trichromatic triangle in T must have two vertices in S and the other not in S. But this is not possible by the definition of shadow-closed set. $\square$

Corollary 1 *If every shadow of a set of triangles T is numerable, then T is untight.*

Proof Take a segment AB in the plane and define the following sets:

$$C_1 = Sh(AB)\ ,\ C_i = C_{i-1} \cup \bigcup_{w_1,w_2 \in C_{i-1}} Sh(w_1 w_2)\ ,\ S = \bigcup_{i=1}^{\infty} C_i\ .$$

Clearly S is numerable and therefore it is a shadow-closed set. $\square$

By Theorem 1 every set of triangles T having non-empty shadows is almost tight. This means that there are trichromatic triangles arbitrarily close to a triangle in T. So, we can suspect that if the set of triangles has some property of "stability" under small movements then it will be tight. We say that a set of triangles is *stable* if for every segment AB on the plane there exist $C \in Sh\,(AB)$ and $\varepsilon > 0$ such that $C \in Sh\,(A_0B_0)$ for every $A_o \in Ball_\varepsilon\,(A)$ and $B_o \in Ball_\varepsilon\,(B)$.

The following is a general criteria for tightness.

Theorem 4 *Every stable set of triangles is tight.*

Proof Let T be a stable set. Clearly, every shadow of T is non-empty and therefore T is almost tight. Let A and B be two points which satisfy the requirements guaranteed by Theorem 2. Since T is stable then there exist $C \in Sh\,(AB)$ and $\varepsilon > 0$ such that $C \in Sh\,(A_0B_0)$ for every $A_o \in Ball_\varepsilon\,(A)$

and $B_o \in Ball_\varepsilon(B)$. On the other hand almost tightness states that there exist $A' \in Ball_\varepsilon(A)$ and $B' \in Ball_\varepsilon(B)$ such that $A'B'C$ is trichromatic. Finally $A'B'C$ is a triangle in T since $C \in Sh(A'B')$. Therefore T is tight. $\square$

If T is a set of triangles such that $\varphi(T) = T$ for every similarity φ then we will say that T is *closed under similarities*. The set of all triangles similar to a given triangle has this property and is untight by corollary 1. However, if shadows have non-empty interior, then the set must be tight.

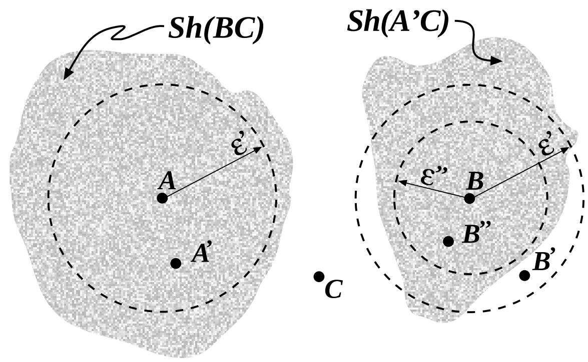
Theorem 5 *If a set of triangles is closed under similarities and all its shadows have non-empty interior, then it is tight.*

Proof Let T be a set of triangles closed under similarities such that every shadow has non-empty interior. We shall prove that T is stable. Let AB be a segment. Since $Sh(AB)$ has non-empty interior, then there exist $C \in Sh(AB)$ and $r > 0$ such that $Ball_r(C) \subseteq Sh(AB)$. Let ε be a positive real number, $A' \in Ball_\varepsilon(A)$, $B' \in Ball_\varepsilon(B)$ and φ the similarity such that $\varphi(A) = A'$, $\varphi(B) = B'$. Denote by C' and r' the point and the number such that $\varphi(Ball_r(C)) = Ball_{r'}(C')$. We have $\lim_{\varepsilon \to 0} C' = C$ and $\lim_{\varepsilon \to 0} r' = r$, so for a sufficiently small fixed ε we obtain that $C \in Ball_{r'}(C') \subseteq Sh(A'B')$ and therefore T is stable. $\square$

We remark that if for a set of triangles, every shadow has non-empty interior, then is not necessarily tight as can be seen from the following example. Take two open disjoint balls in the plane. Color them with two different colors and color the rest of the plane with a third color. Taking the set of all triangles which are not trichromatic in this coloring we observe that it is untight and the shadow of every segment has non-empty interior. However, if shadows are open sets then the set of triangles must be tight.

Theorem 6 *If every shadow of a set of triangles T is open, then T is tight.*

Proof Let us consider an arbitrary 3-coloring of the plane. By Theorem 1 we know that there is an almost trichromatic triangle ABC in the set T . Since $Sh(BC)$ is an open set, there exist $\varepsilon' > 0$ such that $Ball_{\varepsilon'}(A) \subseteq Sh(BC)$. So, by Theorem 2, there exist $A' \in Ball_{\varepsilon'}(A)$ and $B' \in Ball_{\varepsilon'}(B)$ such that $A'B'C$ is trichromatic. Since $A' \in Ball_{\varepsilon'}(A) \subseteq Sh(BC)$, we have that the triangle $A'BC$ is in T. Because $Sh(A'C)$ is open, there exist $\varepsilon'' > 0$ such that $Ball_{\varepsilon''}(B) \subseteq Sh(A'C)$ (see figure 2).



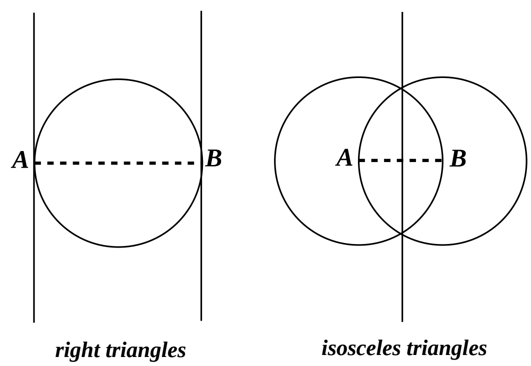
Let B'' be a point in $Ball_{min(\varepsilon'',\varepsilon')}(B)$. Again, by Theorem 2, the point B'' can be chosen of the same color as B' and therefore the triangle $A'B''C$ is trichromatic and belongs to T. We conclude that T is tight. □

Note that there exist sets of triangles with open non-empty shadows which are not stable.

4 Sets of triangles with "thin" shadows

In the preceding section we proved some theorems showing that families with sufficiently "fat" shadows are tight. For example, the set of triangles with an angle in a closed interval is tight by Theorem 5 and the set of triangles with area greater than a given number is tight by Theorem 6.

However, we can not apply those theorems in the case when T is the set of all right triangles or the set of all isosceles triangles. The point is that here the shadows have empty interiors (see figure 3).

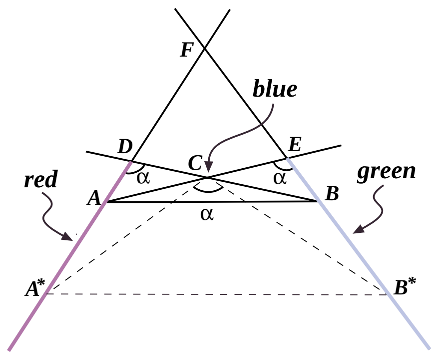


In this section we shall prove that several sets of triangles which are interesting from the geometric point of view are tight though they have shadows with empty interior.

It is not difficult to show that the set of all right triangles is tight. More challenging is the general case in which the triangles have a fixed angle. For a real number $\alpha \in (0, \pi)$ an *α-angle triangle* is a triangle having one of its angles α.

Theorem 7 *The set of α-angle triangles is tight for every $\alpha \in (0, \pi)$.*

Proof Let us start by considering a trichromatic triangle ABC such that $\widehat{BCA} > \alpha$ (the existence of such triangle is granted by Theorem 6). Suppose A, B and C are red, green and blue respectively. Let D and E denote points on the rays $\overrightarrow{BC}$ and $\overrightarrow{AC}$ respectively, such that $\widehat{BDA} = \widehat{BEA} = \alpha$. If D is blue or green then ACD or ABD would become a trichromatic α-triangle, thus we will assume D is red, and by the same reason E is green. If any point X on the ray $\overrightarrow{DA}$ is green or blue then either XDC or XDB would be a trichromatic triangle with an angle α. So, every point on the ray $\overrightarrow{DA}$ is red and by the same reason the whole ray $\overrightarrow{EB}$ is green.



Let F denote the intersection of the lines AD and BE (notice that we may assume AD is not parallel to BE by a suitable choice on the initial triangle). If F happens to be the intersection of the rays $\overrightarrow{EB}$ and $\overrightarrow{DA}$ then we are already done, otherwise F is such that $\widehat{BFA} < \alpha$ (see figure 4). By "moving" A' and B' over the rays $\overrightarrow{DA}$ and $\overrightarrow{EB}$ in such a way that $A'B'$ increases its length and remains parallel to AB we find that the angle $\widehat{B'CA'}$ decreases continuously, having as its limit value the angle $\widehat{BFA}$, but as $\widehat{BFA} < \alpha$ and $\widehat{BCA} > \alpha$, we may assert by the intermediate value Theorem, that there exist $A^* \in \overrightarrow{DA}$ and $B^* \in \overrightarrow{EB}$ such that $\widehat{B^*CA^*} = \alpha$, thus obtaining the desired trichromatic triangle. □

Now, we will deal with sets of isosceles triangles. First of all, let us point out that the family of all isosceles triangles is tight; this can be easily seen by considering the circumcenter of an arbitrary trichromatic triangle. In fact there are several subsets of the set of isosceles triangles which are also tight. The following theorems refer to some of them.

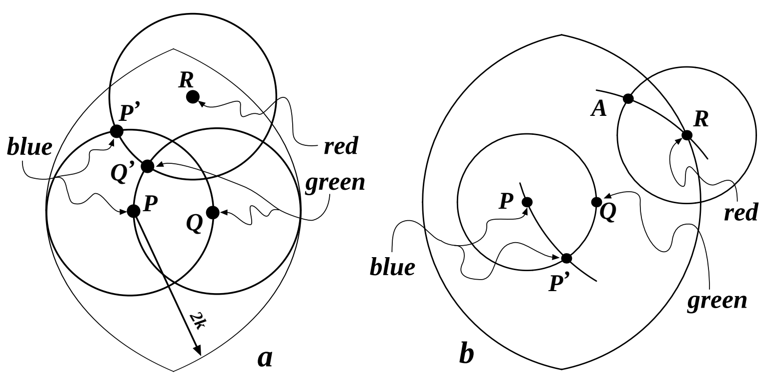
Lemma 1 *For every r-coloring of the plane ($r > 1$), there always exist two points of different colors at a given distance apart.*

Proof Let k be the given distance and A and B be points with different colors such that the length of AB is less than k. Consider a point C such that $AC = BC = k$ and note that either AC or BC is bichromatic in spite of the C color. □

Theorem 8 *The family of isosceles triangles with a side (any of them) of fixed length is tight.*

Proof Let k be an arbitrary positive number, consider an arbitrary blue, red, green coloring of the plane. By the above lemma, let P and Q be points at distance k and assume P is blue and Q is green. Let $F = Ball_{2k}(P) \cap Ball_{2k}(Q)$. Consider the following cases.

1.-There is a red point R in the interior of F. Consider the circles $C_k(R)$ and $C_k(P)$ and denote by P' a common point of these circles which does not lie on the line PQ (note that this point exists because $R \in int\,(F)$).



If P' is colored red or green then $PP'Q$ or $PP'R$ would be the required triangle, therefore P' is blue, and by the same reason, the analogously defined point Q' is green (see figure 5a). Finally notice that the triangle $P'RQ'$ is trichromatic, isosceles and with two sides of length k.

2.- There are no red points on the interior of F. Let us denote $F_r = Ball_r(P) \cap Ball_r(Q)$, $S = \{r \in \mathbf{R}^+ : F_r \text{ contains red points}\}$ and s the infimum of S. We have $\forall \varepsilon > 0 \quad \exists t > s$ such that $t - s \leq \varepsilon$ and there are red points on the F_t boundary. Denote by R a red point on the boundary of F_t. If R is at equal distance to P and Q we are done. So suppose that Q is closer to R than P. Hence $C_k(R) \cap C_{QR}(Q) \cap int\,(F_t) \neq \emptyset$. Take a point X in the above intersection and note that since ε is arbitrarily small we can suppose that X is not red. The triangle XRQ shows that X is not blue, so X is green. All other points in $C_k(R) \cap int\,(F_t)$ must also be green (with the possible exception of points closer to the boundary of F_t than ε). Let us denote by P' the nearest point to Q in $C_k(P) \cap C_{PR}(R)$ and by A the point in $C_k(R) \cap C_{P'R}(P') \cap int\,(F_s)$ (see figure 5b). Using the triangles $PP'Q$ and $PP'R$ we observe that P' must be blue. We can choose ε such that the distance from A to the boundary of F_t is greater than ε and therefore by the previous argument A is green. Hence, the triangle $P'RA$ is isosceles, trichromatic, and with a side of length k .

□

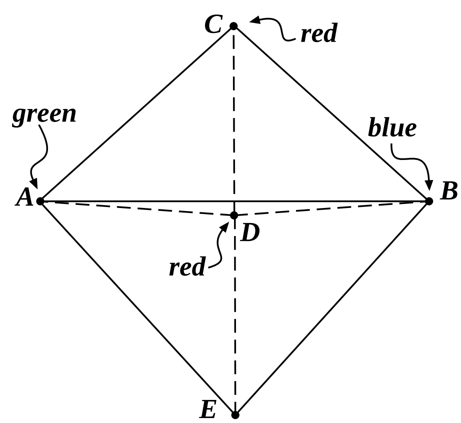
If we strengthen the conditions and ask for the family of isosceles triangles with both equal sides of fixed length then the result is false, this happens because the shadow of every sufficiently large segment is empty; the same holds for the family of isosceles triangles with the "different" side of fixed length, this time by considering the shadow of a sufficiently short segment.

From now on, we denote by ε an arbitrarily small (but fixed) positive real number. An isosceles almost α-triangle is by definition an isosceles triangle such that the angle between the two equal sides belongs to the open interval $(\alpha - \varepsilon, \alpha + \varepsilon)$.

Theorem 9 *If $\alpha = \frac{2\pi}{3}$ or $\frac{\pi}{2}$ then the set isosceles almost α-triangles is tight.*

Proof Let $\alpha = \frac{2\pi}{3}$. Since the set of all equilateral triangles is almost tight then there exists a trichromatic triangle ABC such that all its three angles differ from $\frac{\pi}{3}$ in less than $\frac{\varepsilon}{2}$. Consider the circumcenter D of of the circumscribed circle of ABC and note that $\frac{\widehat{CDB}}{\widehat{CAB}} = \frac{\widehat{ADC}}{\widehat{ABC}} = \frac{\widehat{BDA}}{\widehat{BCA}} = 2$ i.e. the three triangles with vertex D are isosceles almost $\frac{2\pi}{3}$-triangles. Therefore, no matter what color does D have, one of them would be trichromatic, having the desired properties.

Now for the second part, let $\alpha = \frac{\pi}{2}$. By Theorem 1 there is a triangle ABC as shown in figure 6 such that its angles differ from the angles of an isosceles right triangle in less than $\frac{\varepsilon}{2}$. Consider the circumcenter D of ABC. By the same argument as in the first part, triangles ACD and BCD are isosceles almost $\frac{\pi}{2}$-triangles and $\left|\widehat{BDA} - \pi\right| < \varepsilon$. So, we assume that D is red.



Let E denote the point on the bisector of $\widehat{BDA}$ such that $DE = CD$.Observe that $\widehat{EDA} = \widehat{BDE} = \frac{\widehat{BDA}}{2} = \widehat{BCA} = \pi - \widehat{AEB}$. Therefore BDE , DAE and ABE are isosceles almost $\frac{\pi}{2}$-triangles and regardless the color of E we have the desired trichromatic triangle □

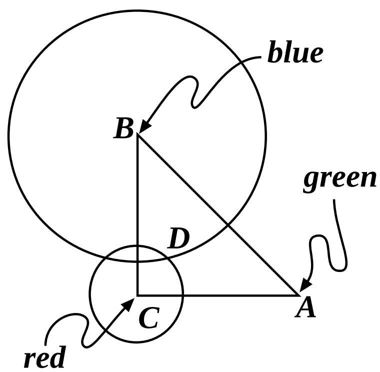
The following result is an application of the above Theorem.

Theorem 10 *If $k \in \left(\sqrt{2}-1, 1\right]$ then the set of triangles with a given ratio k between the lengths of two of their sides is tight.*

Proof If $k = 1$ then we have the set of isosceles triangles, so we may assume that $k \in \left(\sqrt{2}-1, 1\right)$. Consider a trichromatic isosceles almost $\frac{\pi}{2}$- triangle ABC as shown in figure 7 and assume that $CA = CB = 1$.

Let $C_k(C)$ denote the circle with center C and radius k. Let X be a point on $C_k(C)$ and not on the lines BC and AC, if X is blue or green then any of CAX or CBX is a trichromatic triangle with the required ratio between two of its sides. Thus, we may assume that every point on $C_k(C)$ (with the possible exception of four points) is red.

Now consider the circle $C_{k \cdot AB}(B)$, note that $AB \simeq \sqrt{2}$ and since $k > \sqrt{2} - 1$ we have $k + k \cdot AB \simeq k + \sqrt{2}k > 1 = BC$



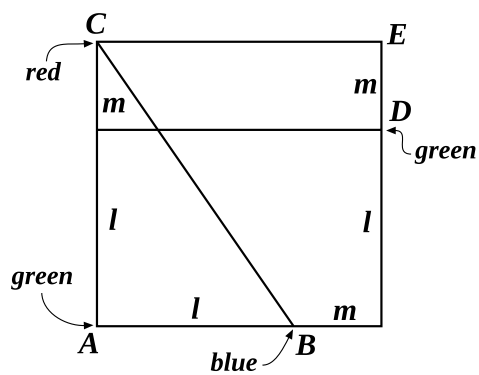
i.e. (see figure 7) the circles $C_k(C)$ and $C_{k \cdot AB}(B)$ intersect each other in a red point D which is not in the line CB since $k < 1$. This allows us to affirm that the triangle ABD is trichromatic and has the given ratio $\frac{BD}{AB} = k$. $\square$

Observe that, if we start the proof with an isosceles almost equilateral trichromatic triangle then the result will hold for any possible k. Actually an interesting open problem is to determine if for any angle $0 < \alpha < \pi$ the set of isosceles almost α-triangles is tight.

Another nice conjecture is that the set of all triangles of fixed area is tight. This problem seems to be difficult and we shall conclude with a much more simpler but somehow similar result.

Theorem 11 *The set of triangles with one of its sides equal to its corresponding height is tight.*

Proof Call a triangle *steady* if it has the property stated in the Theorem. Let us consider a right trichromatic triangle ABC with hypotenuse CB, say A green, B blue and C red. If ABC is isosceles we have nothing to prove. So, suppose $AB = l$ and $AC = l + m,\ \ m > 0$. Let us construct an square of side $l + m$ as shown in the figure 8.



The point D must be green because the triangles ACD and ABD are steady. Since, the triangles EDB , ECA and ECB are steady then, no matter what the color of E is, we are already done.

□

References

[1] Arocha J. L., Bracho J. and Neumann-Lara V., *On the minimum size of tight hypergraphs*, J. Graph Theory **16**, No. 4 (1992), 319-326.

[2] Arocha J. L., Bracho J. and Neumann-Lara V., *Tight and untight triangulated surfaces*, J. Combinatorial Theory, Series B **63**, No. 4 (1995), 185-199.

[3] Arocha J. L., Bracho J. and Neumann-Lara V., *On tight k-graphs*, in preparation.

[4] Erdős P., Graham R. L., Montgomery P., Rothschild B.L., Spencer J.H. and Straus E.G., *Euclidean Ramsey theorems I*, J. Combinatorial Theory, Series A **14**, (1973), 341-363.

[5] Erdős P., Graham R. L., Montgomery P., Rothschild B.L., Spencer J.H. and Straus E.G., *Euclidean Ramsey theorems II*, in: Infinite and Finite Sets, (A. Hajnal et. al. eds), North Holland (1975), pp. 529–558.

[6] Erdős P., Graham R. L., Montgomery P., Rothschild B.L., Spencer J.H. and Straus E.G., *Euclidean Ramsey theorems III*, in: Infinite and Finite Sets, (A. Hajnal et. al. eds), North Holland (1975), pp. 559–584.

[7] Graham R. L., *Rudiment of Ramsey theory*, CBMS Regional Conference Series in Mathematics, 45 American Mathematical Society, Providence, R.I. (1981).

[8] Graham R. L., *Recent trends in Euclidean Ramsey theory. Trends in discrete mathematics.*, Discrete Math. **136**, No. 1 (1994), 119–127.

Instituto de Matemáticas,
National University of México,
México D. F., 04510
e-mail: arocha@josefina.matem.unam.mx