# The size of minimum 3-trees: Cases 3 and 4 mod 6.

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Abstract. A 3-uniform hypergraph is called a minimum 3-tree if for any 3-coloring of its vertex set there is a heterochromatic edge and the hypergraph has the minimum possible number of edges. Here we show that the number of edges in such 3-tree is $\left\lceil \frac{n(n-2)}{3} \right\rceil$ for any number of vertices $n \equiv 3,4 \pmod 6$.

## 1. Introduction

A 3-*graph* is an ordered pair of sets $G = (V, \Delta)$. The elements of $V$ are called *vertices*. The elements of $\Delta$ are subsets of vertices of cardinality 3 and are called *triples*. Given a 3-graph $G = (V, \Delta)$ and a vertex $v$, the *trace* $Tr_G(v)$ of $v$ in $G$ is the graph with vertex set $V \setminus \{v\}$ and a pair $\{x, y\}$ is an edge of $Tr_G(v)$ if and only if $\{v, x, y\}$ is a triple of $G$ . Henceforth, the number of vertices in a 3-graph will be denoted by $n$.

A 3-graph is called *tight* (see [2]) if any proper 3-partition (3-coloring) of the vertex set has a transversal (heterochromatic) triple. A tight 3-graph is called 3-*tree* if whenever we delete a triple from it we obtain an untight 3-graph . Different 3-trees on $n$ vertices may have different number of triples. From the results of [4] we know that the maximum number of triples in any 3-tree is $\binom{n-1}{2}$ . It is not difficult to show that the minimum number of triples in such 3-tree is not less than $\left\lceil \frac{n(n-2)}{3} \right\rceil$. In [2] it was proved that this bound is sharp for any $n$ of the form $\frac{p-1}{2}$ where $p$ is a prime number and it was conjectured that the bound is sharp for any $n$.

Recently, we found that this conjecture is older. In [7] Sterboul explicitly states it. Moreover, he claims that in [6] he gave a proof of the conjecture for the cases $n \equiv 0, 2 \pmod 3$ . Actually he succeed in the construction of 3-graphs which have the property that the trace of any vertex is a tree. Unfortunately, his proofs that those 3-graphs are tight seem to be incorrect.

A 3-graph is called *3-chain* if the trace of every vertex is isomorphic to a chain. It is easy to see that 3-chains are in one to one correspondence with triangular embeddings of complete graphs into surfaces with boundary and the embeddings are such that any vertex lies in the boundary. Reference [5] is classic on triangular embeddings.

In this paper we give constructions of tight 3-graphs for any $n \equiv 3, 4 \pmod 6$ and therefore we prove the conjecture in [2] for those cases. In fact the construction for the case $n \equiv 3 \pmod 6$ gives a family of tight 3-chains. The boundary of those triangulations will be given. It turns out that for $n \equiv 3, 15 \pmod{18}$ this boundary is connected and a tight 3-cycle (see [3]) can be obtained by adding a cone of triangles with center in a new vertex. This proves the conjecture for the minimum number of triples in a 2-tight 3-graph (see [3])for any $n \equiv 4, 16 \pmod{18}$.

The construction for the case $n \equiv 4 \pmod 6$ is obtained by a combinatorial construction from the previous case. Here the trace of one vertex is a cycle and the trace of any other vertex is a chain and the boundary is always connected. So, triangular embeddings of $K_{n+1}$ minus an edge into non orientable closed surfaces can be obtained by adding a cone.

## 2. The case $n \equiv 3 \pmod 6$.

Let us consider the abelian group $\mathbb{Z}_3 \oplus \mathbb{Z}_t$. Its elements are the vertices of our 3-graphs. So, vertices are ordered pairs $(a, x)$. When no confusion arises, we will write $ax$ to denote the vertex $(a, x)$. We only consider the case when $t$ is an odd number and therefore $n = |\mathbb{Z}_3 \oplus \mathbb{Z}_t| = 3t \equiv 3 \pmod 6$.

We will use $a$ to denote an element of $\mathbb{Z}_3$ and $x$ or $y$ to denote elements of $\mathbb{Z}_t$. The symbol + will denote the sum in the appropriate group, namely $\mathbb{Z}_3$ , $\mathbb{Z}_t$ , or $\mathbb{Z}_3 \oplus \mathbb{Z}_t$.

Of course, we know how to add vertices. We can also multiply a vertex by a natural number. If $e = \{v_1, v_2, v_3\}$ is a triple of vertices and $v$ is a vertex then $e + v = \{v_1 + v, v_2 + v, v_3 + v\}$. If $F$ is any set of triples, then $F + v$ is $\{f + v | f \in F\}$. We will say that a set of triples $F$ is *closed* if for any vertex $v \in \mathbb{Z}_3 \oplus \mathbb{Z}_t$ we have $F = F + v$.

Let us consider the following 3 sets of triples:

$$\Delta_t^1 = \{\{0x, 1x, 2x\} \mid x \in \mathbb{Z}_t\}$$
$$\Delta_t^2 = \left\{\left\{ax, ay, \left(a+1, \frac{x+y}{2}\right)\right\} \mid \begin{array}{c} a \in \mathbb{Z}_3, \\ x, y \in \mathbb{Z}_t \end{array}\right\}$$
$$\Delta_t^3 = \left\{\left\{ax, ay, \left(a+1, \frac{x+y+1}{2}\right)\right\} \mid \begin{array}{c} a \in \mathbb{Z}_3, \\ x, y \in \mathbb{Z}_t \end{array}\right\}$$

Observe that $\Delta_t^2$ and $\Delta_t^3$ are well defined since $t$ is odd. We remark that $\Delta_t^1 \cup \Delta_t^2$ is the Steiner Triple System constructed via Skolem's method (see[1] , page 178). Let us consider the 3-graph $G_t = \{\mathbb{Z}_3 \oplus \mathbb{Z}_t, \Delta_t^1 \cup \Delta_t^2 \cup \Delta_t^3\}$. Our purpose is to show that $G_t$ is a tight 3-chain.

The following facts are not hard to prove :

**Proposition 1.** *The sets* $\Delta_t^1$ ,$\Delta_t^2$ *and* $\Delta_t^3$ *are closed.*

**Proposition 2.** *The number of triples in* $G_t$ *is* $\frac{n(n-2)}{3}$. *Moreover* $\Delta_t^1$ ,$\Delta_t^2$ *and* $\Delta_t^3$ *do not intersect pairwise and* $|\Delta_t^1| = t$ , $|\Delta_t^2| = |\Delta_t^3| = 3\binom{t}{2}$.

**Proposition 3.** *The number of triples in* $G_t$ *containing a given pair of vertices is 1 or 2.*

From proposition 1 we have that the group of automorphisms of $G_t$ contains $\mathbb{Z}_3 \oplus \mathbb{Z}_t$ and therefore it acts transitively on the set of vertices of $G_t$. The fact that all vertices are alike will strongly be used below.

**Proposition 4.** *For any* $t$ , $G_t$ *is a 3-chain.*

**Proof** Let $v$ be a vertex of $G_t$. By theorem 1 below and the basic lemma from [2] the trace $Tr_{G_t}(v)$ is a connected graph. By proposition 2 and an standard counting argument $Tr_{G_t}(v)$ has not cycles. By proposition 3 it must be a chain. □

We remark that proposition 4 will not be used until we finish proving theorem 1.

Our task will be easier if we introduce some terminology. For any $x \in \mathbb{Z}_t$ a set of vertices $\{0x, 1x, 2x\}$ will be called a *column*. Note that $\Delta_t^1$ is the set of columns. In a column there is a natural cyclic order induced by the order $0 < 1 < 2 < 0$ of $\mathbb{Z}_3$. On the other hand, for any $a \in \mathbb{Z}_3$ a set of vertices $\{a0, a1, ..., (a, t-1)\}$ will be called a *row*. In a row there is a natural cyclic order induced by the order $0 < 1 < 2 < ... < t-1 < 0$ of $\mathbb{Z}_t$. Due to the toroidal structure (see figure 1) of $\mathbb{Z}_3 \oplus \mathbb{Z}_t$, it makes sense to speak about the next (previous) column or row.

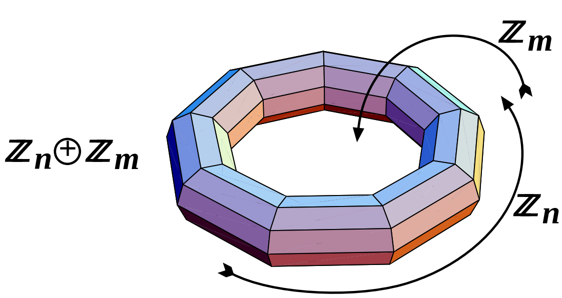


Figure 1. The $\mathbb{Z}_n \oplus \mathbb{Z}_m$ torus.

Let us analyze the trace of one vertex in more detail, say the trace of 00. By definition, this trace is a graph whose vertex set is $\mathbb{Z}_3 \oplus \mathbb{Z}_t \setminus \{00\}$, and whose edge set is the union of the following sets:

$$\Delta_t^1(00) = \{\{10, 20\}\}$$

$$\Delta_t^2(00) = \left\{\left\{0x, \left(1, \frac{x}{2}\right)\right\}, \{2x, (2, -x)\} \;\mid x \in \mathbb{Z}_t\right\}$$

$$\Delta_t^3(00) = \left\{\left\{0x, \left(1, \frac{x+1}{2}\right)\right\}, \{2x, (2, -x-1)\} \;\mid x \in \mathbb{Z}_t\right\}$$

Observe that the trace of 00 restricted to row 2 is the chain 0,-1,1,-2,2,-3,..... in which we have omitted the first coordinate (equal to 2) of each vertex. Since all vertices are alike we have proved the following.

**Proposition 5.** *The trace of a vertex in a row $i$ restricted to the row $i-1$ is a chain.*

Observe that the vertex $(1, \frac{1}{2})$ is of valence 1 in the trace of 00. Since all vertices are alike we have also proved the following.

**Proposition 6.** *An edge is in the boundary of $G_t$ if and only if it is of the form $\{v, v + (1, \frac{1}{2})\}$. The connected components of the boundary are the elements of the quotient group $\mathbb{Z}_3 \oplus \mathbb{Z}_t / < (1, \frac{1}{2}) >$. If $t \equiv 0 \pmod 3$ the boundary has 3 components, otherwise the boundary is connected.*

**Theorem 1.** *$G_t$ is a tight 3-graph.*

**Proof** Suppose it is not true. Let $\{R, B, Y\}$ be a proper red–blue–yellow coloring of $\mathbb{Z}_3 \oplus \mathbb{Z}_t$ having no transversal triple in $G_t$. We shall use the notation $a : \ldots \overset{x}{R} \ldots$ to express that in the row $a$ the $x$-th vertex is in $R$ (i.e. it is colored red).

The "triangle" of colors has two possible orientations: $R \to B \to Y \to R$ or $R \leftarrow B \leftarrow Y \leftarrow R$. We will show that the cyclic order in a trichromatic row is compatible with one of those two orientations. Indeed, suppose that we have the coloring $a : \ldots \overset{x}{R}\overset{x+1}{B} \ldots \overset{y}{R}\overset{y+1}{Y} \ldots$, and denote by $\Theta$ the vertex $\left(a+1, \frac{x+y+2}{2}\right)$. Since $\{ax, (a, y+1), \Theta\}$ , $\{ay, (a, x+1), \Theta\}$ and $\{(a, x+1), (a, y+1), \Theta\}$ are triples in $G_t$ (see figure 2)

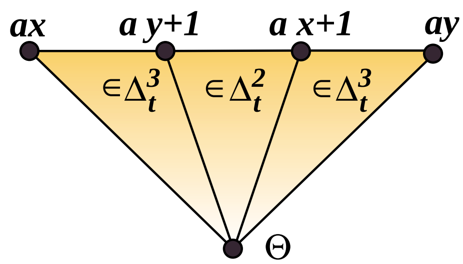


Figure 2. Rows are oriented.

then regardless of the color of $\Theta$ we have a transversal triple. Hence, we can properly say that trichromatic rows have orientations.

Let $a$ be a trichromatic row. Let us prove that any color (say $R$) is in the next row. We have the coloring $a$ : ... $\overset{x}{R}$ ... $\overset{y}{B}\overset{y+1}{Y}$ ... . Denote by $\Theta$ the vertex $\left(a+1, \frac{x+y+1}{2}\right)$. Since $\{ax, ay, \Theta\}$ and $\{ax, (a, y+1), \Theta\}$ are triples in $G_t$ (see figure 3)

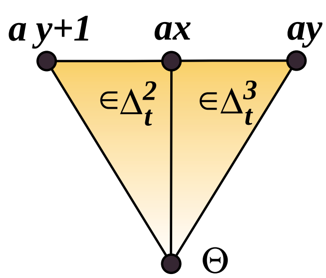


Figure 3. Trichromatic row case

then $\Theta$ must be red. So, if a row is trichromatic then any row is also trichromatic.

If all rows are monochromatic, then each triple in $\Delta_t^1$ is transversal, which is impossible.

If a row is bicolored, then by proposition 5, in the next row no vertex is colored with the third color.

Suppose that there is a bicolored row $a$ say red-blue. In the row $a+1$ there are no yellow vertices. If $a+1$ is a red-blue row, then $a+2$ has no yellow vertices and this contradicts the fact that the coloring is proper. Therefore, we may suppose that $a+1$ is monochromatic red and $a+2$ has yellow vertices. If $a+2$ is monochromatic yellow then any column with a blue vertex is transversal. If $a+2$ is bicolored, then regardless of the second color in it, the third color is present in the next row. This contradicts proposition 5. So, $a+2$ is trichromatic and therefore all rows are trichromatic.

Since there are three rows and two possible orientations, then there is a row (say the row 0) such that the next one has the same orientation.

If row 0 has no consecutive vertices with the same color then $t \equiv 0 \pmod 3$ and we have the following coloring $0 : \overset{0}{R}\overset{1}{B}\overset{2}{Y}\overset{3}{R}\ \dots\ \overset{-3}{R}\overset{-2}{B}\overset{-1}{Y}$. Since $\{10, 00, (0, -1)\}$ and $\{10, 01, (0, -1)\}$ are triples in $G_t$ the vertex 10 must be yellow and, by similarity of vertices, we have the coloring $1 : \overset{0}{Y}\overset{1}{R}\overset{2}{B}\overset{3}{Y}\ \dots\ \overset{-3}{Y}\overset{-2}{R}\overset{-1}{B}$. By the same argument the third row is colored by the coloring $2 : \overset{0}{B}\overset{1}{Y}\overset{2}{R}\overset{3}{B}\ \dots\ \overset{-3}{B}\overset{-2}{Y}\overset{-1}{R}$ and then any column is transversal.

Hence, the row 0 has a vertex $0y$ such that $(0, y+1)$ is of the same color and $(0, y-1)$ is of different color. In this case we have the coloring $0 : \dots\ \overset{x}{R}\ \overset{x+1}{B}\ \dots\ \overset{y-1}{B}\ \overset{y}{Y}\ \overset{y+1}{Y}\ \dots$ .

Denote by $\Theta$ the vertex $\left(1, \frac{x+y}{2}\right)$. Since the triples $\{(0, x+1), (0, y+1), \Theta + 01\}$, $\{0x, (0, y+1), \Theta + 01\}$, $\{0x, (0, y-1), \Theta\}$ and $\{0x, 0y, \Theta\}$, are in $G_t$ (see figure 4),

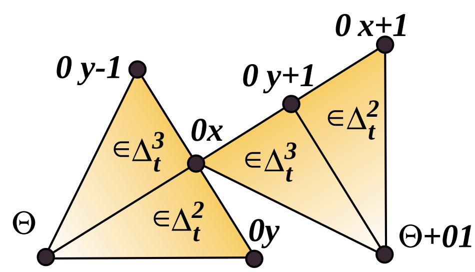


Figure 4. The contradiction.

then $\Theta$ is red and the vertex $\Theta + 01$ is yellow. This contradicts the assumption that row 1 has the same orientation as row 0. □

## 3. An operation on 3-chains.

Let $M$ be a 3-chain with $n$ vertices. An easy counting argument gives that $n$ can not be congruent with $1 \pmod 3$. On the other hand, due to the efforts of Ringel, Young, and others (see [5]) we know that for any $n$ congruent with $0, 2 \pmod 3$ there are 3-chains with $n$ vertices.

When $n \equiv 1 \pmod 3$ the bound for the number of triples in a tight 3-graph is $\frac{n(n-2)+1}{3}$. This bound can be reached in a 3-graph in which the trace of one vertex is a cycle and the trace of any other vertex is a chain. Such 3-graph will be called *almost 3-chain.*

Let $M$ be a 3-chain with $n$ vertices and $n \equiv 0 \pmod 3$ . If $M$ has $\frac{n}{3}$ disjoint triples with no edges in the boundary and $M$ has connected boundary, then there is an easy way to obtain an almost 3-chain $\widetilde{M}$ with $n+1$ vertices namely by gluing a cone of triples with center in a new vertex to the boundary of $M$ (see figure 5) and deleting the set of disjoint triples. Observe that it is not known if the tightness of $M$ implies the tightness of $\widetilde{M}$.

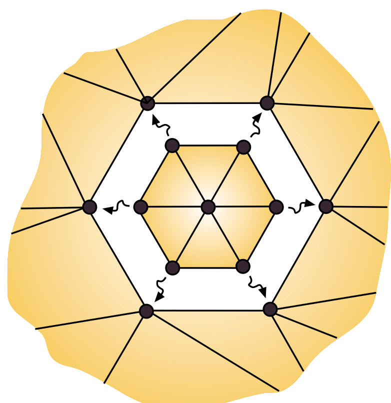

Figure 5. Gluing a cone to the boundary

We shall generalize this construction for the case when the boundary of $M$ is not connected.

First, let us delete from $M$ a set $T$ of $\frac{n}{3}$ disjoint triples with no edges in the boundary thus obtaining the 3-graph $M'$ . Observe that $M'$ is no longer a surface but its one skeleton is still a complete graph. Let us denote by $\partial M'$ the *boundary graph* of $M'$ i.e. the graph whose edges are the pairs of vertices of $M'$ that belong to exactly one triple of $M'$.

There are exactly two edges of the boundary of $M$ and two edges of one of the triples in $T$ incident to each vertex of $\partial M'$. So $\partial M'$ is a 4-regular graph. We must find out a Hamiltonian cycle $H$ in $\partial M'$ that satisfies the following condition: If $v$ is a vertex in $H$ and $v^-, v^+$ are the two vertices adjacent to $v$ in $H$ then $v^-$ and $v^+$ are not in the same connected component of the graph $Tr_{M'}(v)$ .

If $\partial M'$ has such a Hamiltonian cycle, then by gluing a cone of triples with center in a new vertex to $H$, we obtain a 3-graph $\widetilde{M}$ and the following proposition holds.

**Proposition 7.** *$\widetilde{M}$ is an almost 3-chain.*

**Proof** Denote by $\alpha$ the new vertex. The trace of $\alpha$ is the cycle $H$. Let $v$ be any other vertex in $\widetilde{M}$. The vertices $v^-$ and $v^+$ are two endpoints of the two chains which are the trace of $v$ in $M'$. Since we joined them by the edges $\{\alpha, v^-\}$ and $\{\alpha, v^+\}$ , the trace of $v$ in $\widetilde{M}$ is a chain. □

In the following section we shall prove that $\widetilde{G_t}$ is tight.

## 4. The case $n \equiv 4 \pmod 6$.

The set $\Delta_t^1$ of all columns is a set of disjoint triples in $G_t$ having no edges in the boundary. Let $H$ be the union over all $x \in \mathbb{Z}_t$ of the chains $2x \to 0x \to 1x \to (2, x + \frac{1}{2})$. Since $\frac{1}{2}$ generates $\mathbb{Z}_t$, by using proposition 6, we obtain that $H$ is a Hamiltonian cycle in $\partial G_t' = \partial(G_t \setminus \Delta_t^1)$. Now we must prove that $H$ satisfies the property stated in the previous section.

**Proposition 8.** *For any vertex $v$ the vertices $v^-$ and $v^+$ defined by the subchain $v^- \to v \to v^+$ of $H$ are in different connected components of the trace of $v$ in $G_t'$.*

**Proof** First, observe that $\mathbb{Z}_t$ has a natural action on $H$ by adding $0\frac{1}{2}$. By proposition 6 and the remark before proposition 5 we have that the trace of $0x$ , $1x$ and $2x$ in $G_t$ are of the form:

$$0x : (1, x + \frac{1}{2}) \to \cdots \to 1x \to 2x \to \cdots \to (2, x - \frac{1}{2}).$$

$$1x : (2, x + \frac{1}{2}) \to \cdots \to 2x \to 0x \to \cdots \to (0, x - \frac{1}{2}).$$

$$2x : (0, x + \frac{1}{2}) \to \cdots \to 0x \to 1x \to \cdots \to (1, x - \frac{1}{2}).$$

Then, we can check the proposition directly. □

**Corollary 1.** *$\widetilde{G_t}$ is an almost 3-chain with connected boundary.*

**Proof** We must prove that the boundary is connected. It is clear that $\partial\widetilde{G_t}$ is the set $\overline{H}$ of edges in $\partial G_t'$ not in $H$. It is not difficult to show that $\overline{H}$ is the union over all $x \in \mathbb{Z}_t$ of the chains $(2, x - \frac{1}{2}) \to 0x \to (1, x + \frac{1}{2}) \to (2, x + \frac{1}{2})$ and it is a Hamiltonian cycle in $\partial G_t'$ for the same reasons that $H$ is such a cycle. □

In figure 6 we show the set of columns, the boundary, and the cycles $H$, and $\overline{H}$ for the case $t = 3$ .

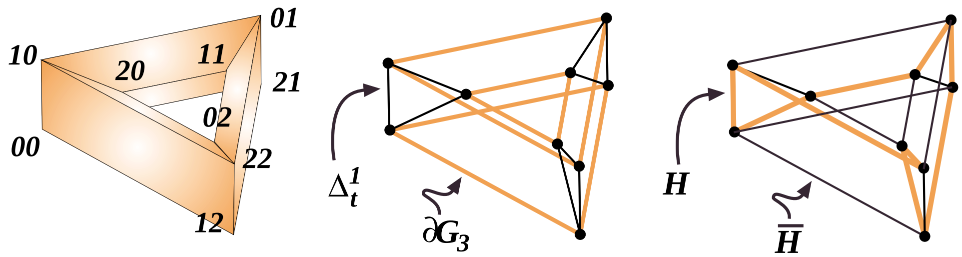


Figure 6. The boundary graph $\partial G_3'$

**Theorem 2.** *$\widetilde{G}_t$ is a tight 3-graph.*

**Proof** The basis for the proof of this theorem is the proof of theorem 1. Let us denote by $\alpha$ the new vertex in $\widetilde{G}_t$. Again, let $\{R, B, Y\}$ be a proper red–blue–yellow coloring of $\mathbb{Z}_3 \oplus \mathbb{Z}_t \cup \alpha$ having no transversal edges in $\widetilde{G}_t$.

If none of the vertices different to $\alpha$ is colored red then $\alpha$ must be colored red. But then some edge $\{v, v'\}$ of $H$ must be bicolored and the triple $\{v, v', \alpha\}$ must be transversal. So, the coloring is proper in $\mathbb{Z}_3 \oplus \mathbb{Z}_t$.

Again tricolored rows have orientation since the triples in $\Delta_t^1$ do not appear in this part of the proof of theorem 1. By the same argument, if a given row is tricolored any row must be tricolored.

If all rows are monochromatic, then the subchain $20 \to 00 \to 10 \to (2, \frac{1}{2})$ of $H$ has any possible bicoloration of an edge, and then, regardless of the color of $\alpha$, we get a tricolored triple.

Observe that proposition 5 also holds for $\widetilde{G}_t$. So, if a row is bicolored, then the next row is not colored with the third color.

Let $a$ be a red-blue bicolored row. Again, we can suppose that $a+1$ is monochromatic red and $a+2$ has yellow vertices. The row $a+2$ can not be bicolored since otherwise the third color would appear in $a$. Suppose that $a+2$ is monochromatic yellow. Let $v$ be a blue vertex in the row $a$ , and let $v^- \to v \to v^+ \to v^{++}$ be the subchain of $H$ containing $v$. Observe that $v^-, v^{++}$ are yellow and $v^+$ is red , and regardless of the color of $\alpha$ we get a tricolored triple. So, $a+2$ is tricolored and therefore all rows are tricolored.

We can suppose that the rows 0 and 1 have the same orientation. If the row 0 has no consecutive vertices with the same color, then as in theorem 1, we conclude that all vertices in $\mathbb{Z}_3 \oplus \mathbb{Z}_t$ are colored in the following way

$$0 : \overset{0}{R}\overset{1}{B}\overset{2}{Y}\overset{3}{R} \dots \overset{-3}{R}\overset{-2}{B}\overset{-1}{Y}$$

$$1: \overset{0}{Y}\overset{1}{R}\overset{2}{B}\overset{3}{Y} \ldots \overset{-3}{Y}\overset{-2}{R}\overset{-1}{B}$$
$$2: \overset{0}{B}\overset{1}{Y}\overset{2}{R}\overset{3}{B} \ldots \overset{-3}{B}\overset{-2}{Y}\overset{-1}{R}$$

and $t \equiv 0 \pmod 3$. Therefore $\{00, 10\}$ , $\{01, 11\}$ and $\{02, 12\}$ are, respectively red-yellow, blue-red, and yellow-blue edges of $H$ and some of them form with $\alpha$ a tricolored triple.

In theorem 1 the case when the row 0 has two consecutive vertices with the same color was proved not using the triples in $\Delta_t^1$. □

## 5. On orientability.

**Proposition 9.** *The surfaces $G_t$ and $\widetilde{G_t}$ are non oriented.*

**Proof** Indeed for any three different element $x, y, z$ in $\mathbb{Z}_t$ , the simplicial path $0x, 0y, 0z, 0x$ reverses orientation in $G_t$ and $\widetilde{G_t}$ . To see this, observe the trace of $0x$ in $G_t$ and $\widetilde{G_t}$ . In figure 7 it is shown how the first coordinates of the vertices of the trace of $0x$ in $G_t$ are ordered.

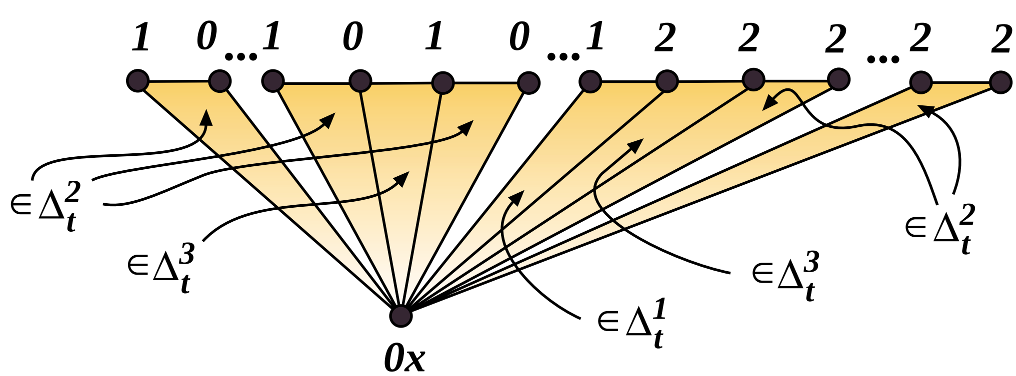


Figure 7. The form of the trace of $0x$ in $G_t$ .

This form of the trace of $0x$ in $G_t$ follows from proposition 5 and the fact that $\Delta_t^2 \cup \Delta_t^1$ is an Steiner Triple System. Since the break triple used to obtain the trace of $0x$ in $\widetilde{G_t}$ is the one in $\Delta_t^1$ , then this trace must have a similar form. From this we have that if $\{0x, 0y, (1, \frac{x+y}{2})\} \in \Delta_t^2$ is given the orientation $(0x, 0y, (1, \frac{x+y}{2}))$ , say, then $\{0x, 0z, (1, \frac{x+z}{2})\} \in \Delta_t^2$ has the orientation $(0x, 0z, (1, \frac{x+z}{2}))$ if $0x$ is to be locally oriented (and thus the trace of $0x$ ). Then, the same argument based on $0y$ and $0z$ would give opposite orientations to $\{0y, 0z, (1, \frac{y+z}{2})\} \in \Delta_t^2$ . □

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