Regular Projective Polyhedra with Planar Faces I

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Abstract

This is the first of two papers in which we classify the regular combinatorial polyhedra which may be immersed in projective 3-space with planar faces and with all of its combinatorial symmetries realized by isometries. In this paper we develop the basic notions and conclude with the case in which the polyhedron is an embedded surface. The main result is a diophantine trigonometric equation relating the combinatorial and geometric parameters of such polyhedra.

0 Introduction

Intuitively, a regular projective polyhedron is a regular combinatorial polyhedron which is drawn in the projective 3-dimensional space, $\mathbb{P}^3$, in such a way that all symmetries are realized by geometric isometries of the ambient space. In this work, we restrict to the case where each face lies on a projective plane, leaving the case of non-planar faces for further studies (in analogy with what Grünbaum [8] and Dress [6, 7] did for polyhedra in $\mathbb{R}^3$). The regular projective polyhedra with planar faces can be subdivided by the property of being embedded as a surface (the "Platonic" case) and when it has self intersections (the "Kepler-Poinsot" case).

For each regular projective polyhedron, its natural double cover in the 3-sphere can be seen as a regular euclidian polyhedron in $\mathbb{R}^4$. And conversely, a

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bounded regular euclidian polyhedron in $\mathbb{R}^4$ projects to a projective one. So that, in principle, these objects are equivalent. However, we found it easier to visualize and to work with them in $\mathbb{P}^3$; and moreover, we think it is the natural approach to the problem. In this perspective, it is interesting to review the well known euclidian 3-dimensional case arising from the projective plane (see Section 2).

In the language of polyhedra in $\mathbb{R}^4$, Coxeter discovered in [2] all the regular polyhedra corresponding to embedded surfaces, and he suggests the existence of some of the Kepler-Poinsot type. On the other hand, in the study of elliptic honeycombs, [3], once again Coxeter takes the projective geometry approach for the study of geometric regular polytopes.

We should also note that in the terms of the article [9] of McMullen, we are classifying the faithful realizations of regular combinatorial polyhedra in $\mathbb{P}^3$; or equivalently in $\mathbb{R}^4$.

In this paper, after establishing general terminology and some basic facts, we obtain a diophantine trigonometric equation [1], called the *waist equation*, which is a necessary condition on some of its combinatorial parameters for the existence of a regular projective polyhedron with planar faces. All the integer solutions of this equation give rise to the skew regular polyhedra of Coxeter, yielding a new proof of his classification Theorem. In a future paper we will see that not all of the rational solutions of the waist equation come from regular projective polyhedra, but they play an important role for the classification.

We should remark that some intrinsically projective constructions developed in the paper give rise naturally to previously unknown polyhedra in $\mathbb{P}^3$, and hence $\mathbb{R}^4$.

1 Definitions

Although some definitions and general arguments may be carried out in the greater generality of incidence polytopes, we shall restrict to polyhedra.

A *combinatorial polyhedron* $\mathcal{P}$ consists of a graph, called its 1-skeleton and denoted $Sk^1(\mathcal{P})$, together with a collection of cycles, called the *faces* of $\mathcal{P}$, satisfying some additional properties stated bellow. First, the 1-skeleton is connected and it has no loops, but it may have multiple edges. For each vertex v define a graph called the *vertex figure* of v, whose vertices are the edges incident to v and making two of them adjacent if there is a face containing them. The second condition is that all the vertex figures are cycles. Observe that this implies that each edge is in two faces, and that these conditions are the combinatorial translation of asking that when the faces are viewed as 2-cells attached to the 1-skeleton one obtains a connected surface.

A *flag* is any incident triplet (vertex, edge, face). A combinatorial polyhedron $\mathcal{P}$ is said to be *regular* if its group of automorphisms, $Aut(\mathcal{P})$, acts transitively on the set of all flags. In particular, this implies that all faces of a regular polyhedron are cycles of the same length, p say, and that all the vertices have the same degree, q say. The ordered pair $\{p, q\}$ is called the Schläfli Symbol of $\mathcal{P}$.

The n-dimensional Projective Space $\mathbb{P}^{\mathrm{n}}$, also known as Elliptic Space, is the n-dimensional sphere S^n with antipodes identified. Its metric, as well as other geometric notions, come from this identification. Thus, its group of isometries is $Iso(\mathbb{P}^{\mathrm{n}}) = \mathrm{PO(n+1)} = \mathrm{O(n+1)}/\{\mathrm{I}, -\mathrm{I}\}$, where I is the identity matrix. Observe that for any pair of points in $\mathbb{P}^{\mathrm{n}}$ there are exactly two line segments joining them, such segments, which together form a line, will be called *opposite*. Given a projective subspace Π of $\mathbb{P}^{\mathrm{n}}$, its *polar space* is the set of points at maximal distance $\pi/2$ from Π; it is also a projective subspace and their dimensions add to $n-1$.

A *projective graph* is a finite set of points in $\mathbb{P}^{\mathrm{n}}$, called vertices, together with a set of line segments, called edges, joining some pairs of these vertices. Clearly, there is an underlying combinatorial graph with no loops and at most double edges. Given a projective graph G, the *opposite* graph G^{op} has the same vertices as G, and for each edge we choose the opposite segment between the corresponding vertices. Thus, G^{op} is combinatorially isomorphic to G, but in general it is a different projective graph. A *linear map* of a graph G to $\mathbb{P}^{\mathrm{n}}$, $g : G \to \mathbb{P}^{\mathrm{n}}$, is a surjective graph homomorphism (in the sense that vertices go to vertices and edges to edges) to a projective graph. Clearly, any linear map has an *opposite* linear map.

A *regular projective polyhedron* $\mathcal{P}$ consists of a regular combinatorial polyhedron $\mathcal{P}_c$ together with a linear map

$$g : Sk^1(\mathcal{P}_c) \to \mathbb{P}^{\mathrm{n}}$$

for which there exists an injective homomorphism

$$\gamma : Aut(\mathcal{P}_c) \to Iso(\mathbb{P}^{\mathrm{n}}) \ ,$$

such that for every $\rho \in Aut(\mathcal{P}_c)$ we have

$$g \circ \rho = \gamma(\rho) \circ g \ .$$

Two regular projective polyhedra are called *equivalent* (and will be regarded as being the same) if their combinatorial polyhedra are isomorphic and if there exists an isometry of $\mathbb{P}^n$ making the linear maps commute with the isomorphism. A regular projective polyhedron $\mathcal{P}$ in $\mathbb{P}^n$ is *degenerate* if its associated projective graph $g(Sk^1(\mathcal{P}_c))$ lies in a non trivial projective subspace, that is, if it can be considered as a projective polyhedron in $\mathbb{P}^k$ with $k < n$.

There is an important class of regular projective polyhedra which we call *Platonic* because they resemble their euclidian analogues. Consider a projective graph in $\mathbb{P}^3$ with a distinguished collection of cycles, each of which lies in a projective plane and bounds there a topological disk. If the union of this closed planar disks is a surface, we have a combinatorial polyhedron with a geometric realization. It is a *Platonic Projective Polyhedron* if it is combinatorially regular and every automorphism may be realized by an isometry of the ambient space. The obvious examples are the platonic solids thinking of projective space as euclidian space plus a plane at infinity. Of course, they will be projectively different according to their "size" or *radius*, which is the distance of a vertex to the center of symmetry, which can take values from 0 to $\pi/2$.

1.1 Relation to Euclidean Polyhedra

We will see how regular projective polyhedra are in natural correspondence to bounded regular euclidean polyhedra one dimension higher.

Given a regular projective polyhedron $\mathcal{P}$ in $\mathbb{P}^n$ as above, we can *lift* it to a regular euclidean polyhedron $\tilde{\mathcal{P}}$ in $\mathbb{R}^{n+1}$ as follows. Let $\pi : S^n \to \mathbb{P}^n$ be the natural double cover. The inverse image under π of the projective graph $G = g(Sk^1(\mathcal{P}_c))$ is a "geodesic" graph $\tilde{G}$ in S^n, and we have a double cover $\pi : \tilde{G} \to G$. Let H be the pullback of π by g, that is, the vertices (and edges) of H are pairs (α, β) with $\alpha \in Sk^1(\mathcal{P}_c)$ and $\beta \in \tilde{G}$ such that $g(\alpha) = \pi(\beta)$. H is a double cover of $Sk^1(\mathcal{P}_c)$ on which we have a natural collection of cycles as follows. For every face of $\mathcal{P}_c$, consider its cycle in $Sk^1(\mathcal{P}_c)$, it lifts to either two isomorphic cycles in H or to a single one of twice its length, declare both of them, or it, as the case may be, as distinguished cycles of H. It is easy to see that these cycles satisfy the properties of being faces of a polyhedron $\tilde{\mathcal{P}}_c$, except, possibly, that it may not be connected (corresponding to whether $\tilde{H}$ and $\tilde{G}$ are not connected); in this case let $\tilde{\mathcal{P}}_c$ be one component. Clearly, $\tilde{\mathcal{P}}_c$ comes equipped with a map of its vertices to $\mathbb{R}^{n+1}$. And moreover, by the construction we have an injective homomorphism $\tilde{\gamma} : Aut(\tilde{\mathcal{P}}_c) \to O(n+1)$ (which covers $\gamma : Aut(\mathcal{P}_c) \to Iso(\mathbb{P}^n)$). These are the ingredients of a regular euclidean polyhedron (in McMullen's terminology, [9], a faithful realization of a regular incidence-polyhedron; see also [6]).

Conversely, consider a bounded regular euclidean polyhedron $\tilde{\mathcal{P}}$ in $\mathbb{R}^{n+1}$, it consists of a regular combinatorial polyhedron $\tilde{\mathcal{P}}_c$, an injective homomorphism $\tilde{\gamma} : Aut(\tilde{\mathcal{P}}_c) \to O(n+1)$ and a compatible non-trivial map of its vertices to $\mathbb{R}^{n+1}$ (see [9] and [6]). Since $\tilde{\mathcal{P}}$ is bounded, then up to a scalar factor we may assume its vertices lie on the unit sphere S^n. Now, the 1-skeleton may be mapped uniquely to a rectilinear graph, then projected out to S^n from the origin and down to $\mathbb{P}^n$ to obtain a linear map to $\mathbb{P}^n$. Two cases must be considered. First, if the antipodal map, $-\mathrm{I}$, is not a symmetry of $\tilde{\mathcal{P}}$, then a regular projective polyhedron $\mathcal{P}$ is obtained by the simple composition with the projection (for example the Tetrahedron). And second, if the antipodal map $-\mathrm{I}$ is a symmetry of $\tilde{\mathcal{P}}$, then define the combinatorial polyhedron $\mathcal{P}_c$ to be the quotient $\tilde{\mathcal{P}}_c/\{\mathrm{I}, -\mathrm{I}\}$ observing that its 1-skeleton and automorphism group map naturally to $\mathbb{P}^n$.

2 Planar Polyhedra

In this section we briefly describe the 18 regular projective polyhedra in $\mathbb{P}^2$, which, according to the previous section, correspond to the 18 bounded regular polyhedra in $\mathbb{R}^3$ (see e.g. [8]).

Consider the five platonic solids in $\mathbb{R}^3$ and project them, as in 1.1, to $\mathbb{P}^2$. Let us denote them $[\![3,3]\!]$, $[\![4,3]\!]$, $[\![3,4]\!]$, $[\![5,3]\!]$ and $[\![3,5]\!]$. Their 1-skeleta are drawn respectively in Figure 1 by stereographic projection to $\mathbb{R}^2$ and thus the boundary is to be antipodally identified.

Observe that on the graph (e) we may take the pentagons that surround each vertex as faces for a new polyhedron, which we denote $[\![5,5/2]\!]$. (The precise meaning of the notation we are using will be given in Lemma 1 of Section 4.1). Now, observe that (a) and (b) are opposite graphs and that (c) is opposite to itself. The opposite graphs of (d) and (e) are respectively (f) and (g) of Figure 2.

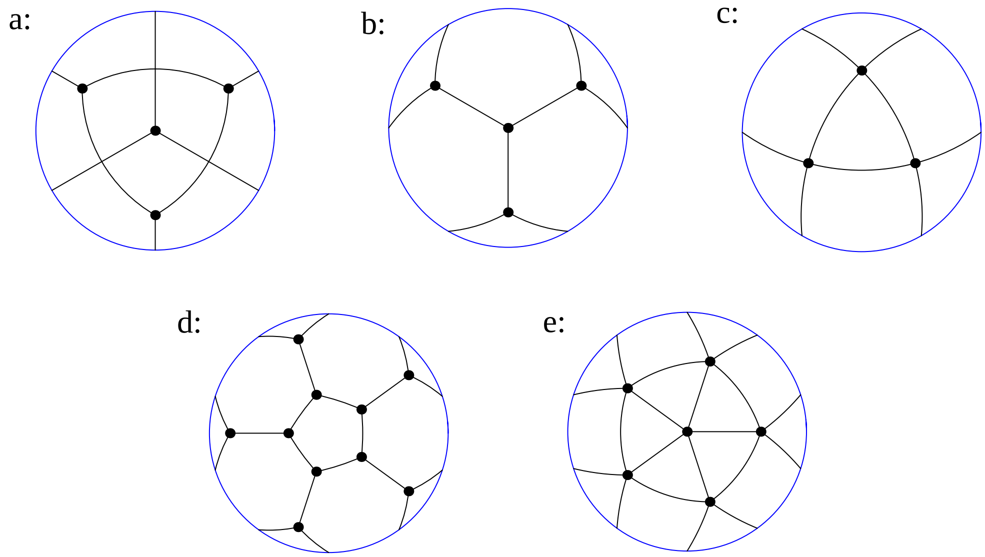

Figure 1: The 1-skeleta of the projected Platonic Solids.

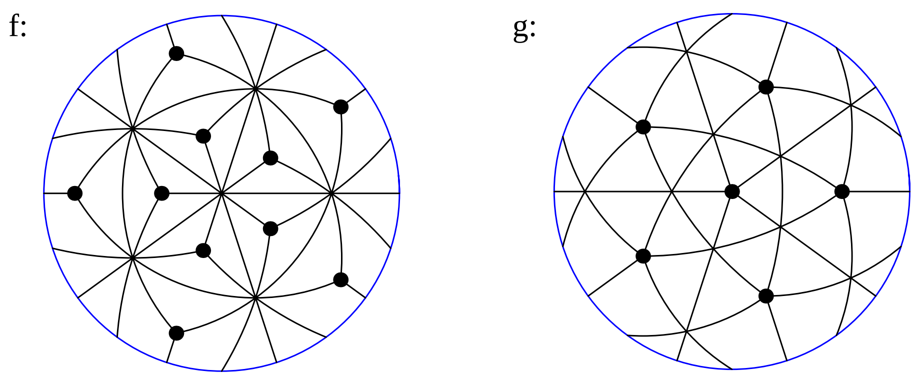

Figure 2: The opposite graphs of $Sk^1[\![5,3]\!]$ and $Sk^1[\![3,5]\!]$.

Consider on the graph (f) the five outmost vertices, they form a regular polygon of type $[\![5/2]\!]$ (a "pentagram"), with all such pentagrams as faces we obtain the polyhedron $[\![5/2,3]\!]$. Finally, on the graph (g) we may take the pentagrams around each vertex to obtain $[\![5/2,5]\!]$, or the big triangles (obtained by fixing one edge and one of its sides then at its ends skip the next edge on the same side and take the following one) to get the polyhedron $[\![3,5/2]\!]$. The last four we have encountered lift to the star and stellated polyhedra of Kepler-Poinsot.

Observe that given any regular projective polyhedron $\mathcal{P}$, we may obtain another one $\mathcal{P}^{op}$ by keeping the same combinatorial and group information but taking the opposite linear map of the 1-skeleton. If we do this on the 9 polyhedra above and then lift them to $\mathbb{R}^3$, we obtain the other 9 which were previously described as Petrie polyhedra. But now they arise in a different order. For example the lift of $[\![3,3]\!]^{op}$ is the Petrie of the Cube, and the lift of $[\![4,3]\!]^{op}$ is the Petrie of the Tetrahedron.

3 Regular Polygons in $\mathbb{P}^3$.

A *polygon* $\mathcal{L}$ in $\mathbb{P}^n$ is a projective graph which is combinatorially a cycle and it is *regular* if there exists a compatible inclusion of its combinatorial automorphisms as isometries of the ambient space. It is *degenerate* if it lies in a non-trivial projective subspace. The simplest regular polygons lie in $\mathbb{P}^1$. They are classified by rational numbers p/q (where in such expressions we always assume p and q are relatively prime) with $p/q > 1$ as follows. Let $[\![p/q]\!]^1$ consist of p successive segments in $\mathbb{P}^1$ of length $(q/p)\pi$. Recall that the length of $\mathbb{P}^1$ is π so that $[\![p/q]\!]^1$ is combinatorially a cycle of length p which winds q times around the projective line.

Let $\mathcal{L}$ be a regular polygon in $\mathbb{P}^n$. If we fix a flag, that is a vertex v and an incident edge e, we obtain canonical generators of the dihedric group $Aut(\mathcal{L})$. Namely ρ_0 and ρ_1 such that ρ_1 fixes the vertex v and transposes its two edges, and ρ_0 fixes e as a segment but transposes its two vertices. Without confusion we may consider ρ_0 and ρ_1 as isometries of $\mathbb{P}^n$, and they satisfy the relations $\rho_0^2 = \rho_1^2 = (\rho_0\rho_1)^p = id$, where p is the length of the cycle and id is the identity of $\mathbb{P}^n$.

3.1 Planar Polygons

In the projective plane $\mathbb{P}^2$, the canonical generators ρ_0 and ρ_1 of a non-degenerate regular polygon $\mathcal{L}$, are reflections along lines ℓ_0 and ℓ_1. These lines meet at a point, called the center of symmetry, at a rational angle of the form $(q/p)\pi$ with $q/p < 1/2$. The distinguished vertex v of $\mathcal{L}$ lies in ℓ_1 at a distance r from the center (with $0 < r < \pi/2$); r is called the *radius*. Finally, the distinguished edge of $\mathcal{L}$ (going from v to $\rho_0(v)$) may cross ℓ_0 orthogonally or it may be the opposite segment that passes through the polar point of ℓ_0. Let us denote the first case by $[\![p/q\,;r]\!]$. It is a projective version of the classic euclidian $\{p/q\}$ and will be called an *inessential regular polygon of type* $[\![p/q]\!]$ (see Figure 3 for a polygon of type $[\![7/3]\!]$). The other case is simply its opposite $[\![p/q\,;r]^{op}$, and will be called *essential* because every line intersects it.

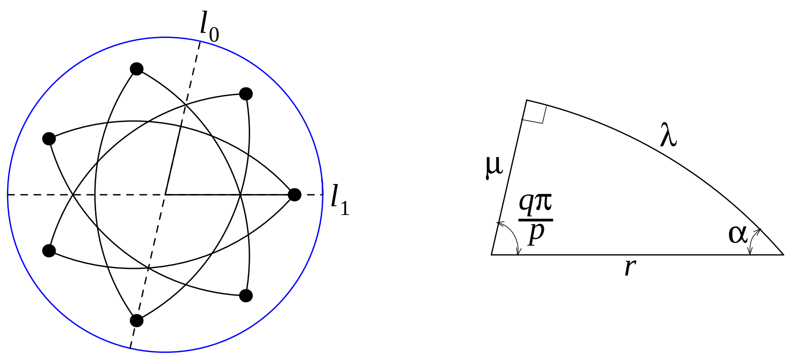

Figure 3: An inessential regular polygon and its basic triangle.

There are other real invariants of $[\![p/q\,;r]\!]$. Namely, the length of the edge 2λ, the internal angle among consecutive sides 2α and the distance of the center to the edge μ (see Figure 3). By the spherical law of cosines they are related by the following equations:

$$\begin{aligned} \cos(q\pi/p) &= \sin(\alpha)\cos(\lambda) \\ \cos(r) &= \cos(\mu)\cos(\lambda)\ . \end{aligned} \tag{1}$$

Observe that $[\![p/q\,;r]\!]$ projects from its center to a regular polygon $[\![p/2q]\!]$[1] in its polar line. This projection is a combinatorial isomorphism or a double cover according to whether p is odd or even.

3.2 Polygons in Projective Space

In projective space $\mathbb{P}^3$ there are two types of non-degenerate regular polygons. They are best characterized by the "dimensions" of their group generators. Let $\rho \in Iso(\mathbb{P}^3)$ be an involution, that is $\rho^2 = id$. Define $dim(\rho)$ to be -1 if ρ has no fixed point, and otherwise the maximum dimension of a pointwise fixed projective subspace. If $dim(\rho) = 2$, it is a reflection along a plane and at the same time an inversion on its polar point. If $dim(\rho) = 1$, it is a π-rotation along a line and also along its polar line. And if $dim(\rho) = -1$, it is a $\pi/2$ translation along a pair of polar lines; however, this case does not arise in our present context because our involutions have fixed points.

Let $\mathcal{L}$ be a regular polygon in $\mathbb{P}^3$ with distinguished flag v, e, and canonical generators ρ_0 and ρ_1. Suppose $dim(\rho_0) = 2$ and let Π_0 be the reflection plane. Consider the plane Π generated by the segments e and $\rho_1(e)$ (which meet at v). Since e is orthogonal to Π_0, so is Π, and therefore ρ_0 and ρ_1 fix Π (as a set, not pointwise). We may conclude that $\mathcal{L}$ lies in Π and so it is degenerate. This leaves us with only two possibilities when $\mathcal{L}$ is non-degenerate: either $dim(\rho_0, \rho_1) = (1, 2)$ and we call it *skew*, or $dim(\rho_0, \rho_1) = (1, 1)$ and we call it a *helicoid*.

3.2.1 Skew Polygons

Consider a skew regular polygon $\mathcal{L}$ as above. Let ℓ_0 be the pointwise fixed line of ρ_0 that meets e at its midpoint, and let ℓ_0' be its polar line. Let Π_1 be the reflection plane of ρ_1 and let $c = \ell_0 \cap \Pi_1$, $c' = \ell_0' \cap \Pi_1$. Let Π be the polar plane of c' and observe that ρ_0 and ρ_1 fix it. Thus, the projection of $\mathcal{L}$ from c' to the plane Π yields a planar regular polygon, called the *symmetry polygon* of $\mathcal{L}$ on its *symmetry plane* Π, which is inessential because ℓ_0 intersects its basic edge. The type $[\![p/q]\!]$ of this planar polygon will be called the *type* of the skew polygon $\mathcal{L}$.

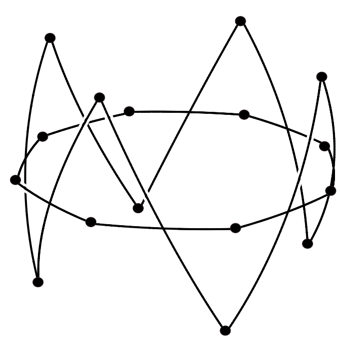

Figure 4: A skew regular polygon of type $[\![8]\!]$ and its symmetry polygon.

Conversely, a skew regular polygon of type $[\![p/q]\!]$ is obtained from a fixed planar polygon $[\![p/q\,;r]\!]$, by moving the vertices alternatively over and under the plane a fixed distance along orthogonal lines, taking the corresponding edges that intersect the plane. If p is even this process results in a skew polygon of p sides called *antiprismatic*. But if p is odd, after one turn we are on the other side of the plane and have to turn once more yielding a polygon of $2p$ sides called skew *prprismaticsmatic*. Corresponding respectively to the case when the projection to the symmetry polygon is a single or a double cover.

Note that we have changed the classic usage of notation. The euclidian analogue of our "skew of type $[\![p/q]\!]$", is the classic prismatic skew polyhedron of type $\{2p/q\}$ (p odd), whose notation is based on the fact that it is combinatorially of length $2p$. However, we have dared to change established notation because it seems to fit better in the general theory, see for example the Uniqueness Principle (Lemma 1, bellow) where no assumption has to be made on the parity. And the rule of translation is simple: skew of type $[\![p/q]\!]$ is prismatic of length $2p$ if p is odd, and antiprismatic of length p if p is even.

Observe finally, that the opposite of a skew regular polygon is again skew, and that for a fixed type there is a two parameter family of geometrically different regular polygons.

3.2.2 Helicoids

Finally, let us describe the helicoids for completeness sake, although we will not consider them in this paper. Suppose $\mathcal{L}$ is a non-degenerate regular polygon with $dim(\rho_0,\rho_1) = (1,1)$. Let ℓ_0 and ℓ_0' be the polar lines about which ρ_0 is a π-rotation, and likewise define ℓ_1 and ℓ_1' for ρ_1. If ℓ_0 and ℓ_1 meet, $\mathcal{L}$ is easily seen to be degenerate, so that if $\mathcal{L}$ is a helicoid no pair of these four lines intersect. Then, there is a pair of polar lines ℓ and ℓ' which intersect orthogonally the previous four. The projection of $\mathcal{L}$ from ℓ to ℓ' and from ℓ' to ℓ give regular polygons in $\mathbb{P}^1$ over which $\mathcal{L}$ may wind around several times. However, they are $[\![p/q]\!]^1$ and $[\![p'/q']\!]^1$. These linear polygons define a *type* of the helicoid, and a real number giving the distance from a vertex to ℓ say, specifies it geometrically.

4 Polyhedra in $\mathbb{P}^3$

4.1 Generalities, definitions and notation

Let $\mathcal{P}$ be a regular polyhedron in $\mathbb{P}^3$ with distinguished flag $v < e < f$, and let ρ_0, ρ_1, ρ_2 be the canonical generators of its automorphism group with respect to this flag (that is, ρ_0 fixes e and f but moves v; ρ_1 fixes v and f but moves e, and ρ_2 fixes v and e but moves f). We will think of ρ_i as an isometry of $\mathbb{P}^3$. The face f is a regular polygon in $\mathbb{P}^3$ with canonical generators ρ_0 and ρ_1. The type of this polygon is the coarsest classification of such polyhedra. Later in this work we will mainly analyse the case of polyhedra with planar faces.

There is another regular polygon associated to $\mathcal{P}$, called its *Vertex Figure*, $VF(\mathcal{P})$, defined combinatorially in Section 1. Its vertices are the barycenters, or midpoints, of the edges in $Sk^1(\mathcal{P})$ incident to v, and two of them are combinatorially adjacent if their edges lie in a common face. To choose the appropriate segment, observe that two adjacent vertices in $VF(\mathcal{P})$ are the endpoints of a path made of two half-edges meeting at v. Choose the segment that makes this triangle homotopically trivial, that is, that defines an ordinary triangle in the plane of the three points.

We claim that $VF(\mathcal{P})$ *is not a helicoid*. Observe that ρ_1 and ρ_2 are the canonical generators of $VF(\mathcal{P})$. Since ρ_1 and ρ_2 fix the vertex v, all the automorphisms do. It is not hard to see that in a helicoid the isometry that corresponds to a generating rotation has no fixed points. Observe also that v is the center of symmetry of the vertex figure and that if it is planar it must be inessential.

We have therefore described $VF(\mathcal{P})$ as a regular polygon which is planar-inessential or skew of type $[\![q_1/q_2]\!] =: [\![q]\!]$, for some rational $q > 2$.

4.2 Planar Faced Regular Polyhedra

Now, suppose f is planar. We may also assume it is inessential, for otherwise we may change $\mathcal{P}$ for its opposite. Then, the face f is $[\![p_1/p_2\,;r]\!] =: [\![p\,;r]\!]$ for some rational $p > 2$ and $0 < r < \pi/2$.

Lemma 1 (Uniqueness principle) *There is at most one regular polyhedron with planar faces $[\![p\,;r]\!]$ and vertex figure of type $[\![q]\!]$. If it exists we call it $[\![p, q\,;r]\!]$.*

Proof. The main idea is that there is at most one way to fit copies of the prescribed face $[\![p\,;r]\!]$ around a vertex v in such a way that the vertex figure turns out to be of type $[\![q]\!]$. The details follow.

Let 2λ be the length of the edge in $[\![p\,;r]\!]$, and let ν be the length of the segment that joins inessentially two barycenters of incident edges (observe that ν must be the length of the edge in the vertex figure). Consider the polygon $[\![q\,;\lambda]\!]$ centered at v on a plane Π', and let $2\lambda'$ be the length of its side.

There are three cases to analyse. First, if $\nu = 2\lambda'$ the vertex figure is planar and lies in the same plane of the face. So that the polyhedron becomes planar. Given p and q it is easy to see that there is at most one way to choose r for this to happen, namely when the internal angle of $[\![p\,;r]\!]$ is equal to $2\pi/q$. Thus we may simplify the notation for the polyhedron and call it $[\![p,q]\!]$ as we did in Section 2.

Second, if $\nu < 2\lambda'$ the vertex figure must be planar and it is obtained by moving simultaneously all the vertices of $[\![q\,;\lambda]\!]$ to the same (local) side of Π' along orthogonal planes passing through them and v and keeping them at distance s from v. In such a process the vertices define planar polygons of type $[\![q]\!]$ and the edge shrinks until we reach ν, that polygon is the vertex figure VF.

And third, if $\nu > 2\lambda'$ the vertex figure must be skew. Analogously, move the vertices alternatively to the two sides of Π' forming skew polygons whose vertices are at distance s from v. In this process the length of the side grows monotonously up to the upper bound 2λ when the vertices cluster at the orthogonal line to Π' at v. If $\nu < 2\lambda$ there is a unique skew polygon of side ν with vertices at distance λ from its center of symmetry, call it VF.

Suppose that the given data (p ,q and r) produces, as above, a vertex figure. Let f be a fixed polygon $[\![p\,;r]\!]$ on a plane Π with distinguished flag $v < e$, and let ℓ_0 , ℓ_1 be its canonical symmetry lines (see Figure 3). Corresponding to the additional data $[\![q]\!]$, construct the vertex figure VF with center v and with distinguished flag corresponding to e and f. Let ρ_2 be its second canonical generator (the reflection along the plane orthogonal to Π' and passing through e). Let ρ_0 be the reflection on the plane orthogonal to Π at ℓ_0. Finally, if VF is planar let ρ_1 be the reflection on the plane orthogonal to Π at ℓ_1, and if VF is skew let ρ_1 be the π-rotation along ℓ_1. By definition, ρ_0 and ρ_1 serve as canonical generators for f, while ρ_1 and ρ_2 are the canonical generators for VF.

Consider the subgroup of $Iso(\mathbb{P}^3)$ generated by ρ_0 , ρ_1 and ρ_2. The Wythoff's construction (c.f. [4], [10]) on this group yields a combinatorial polyhedron, and its action on the vertex v and the segment e gives a linear map of its one skeleton to projective space. This is $[\![p, q\,; r]\!]$. $\square$

A natural question that arises is to classify the $[\![p, q; r]\!]$ which are finite polyhedra. Observe also that from the proof we obtain the following.

Corollary 1 *Let $\mathcal{P}$ be a regular non-degenerate polyhedron in $\mathbb{P}^3$ with planar face, and let ρ_0 , ρ_1 and ρ_2 be canonical generators. Then*

- *$VF(\mathcal{P})$ is planar if and only if $dim(\rho_0, \rho_1, \rho_2) = (2, 2, 2)$.*
- *$VF(\mathcal{P})$ is skew if and only if $dim(\rho_0, \rho_1, \rho_2) = (2, 1, 2)$.*

4.3 Planar-planar polyhedra

Let $\mathcal{P}$ be a non-degenerate regular polyhedron with planar-inessential face and planar vertex figure. By the previous corollary the three canonical generators are reflections on planes and these planes meet at a point c which is fixed by the automorphism group. Let Π be the polar plane of c. The projection of $\mathcal{P}$ from c to Π is a double (or single) cover of a planar polyhedron with inessential faces. And then it is easy to see that $\mathcal{P}$ is a projective embedding of one of the classic nine (Platonic and Kepler-Poinsot) euclidian polyhedra, which depends on a radius parameter (the distance from a vertex to c).

5 Planar-skew polyhedra

From now on we shall assume that $\mathcal{P}$ is a regular polyhedron in $\mathbb{P}^3$ with planar-inessential face and skew-antiprismatic vertex figure. Then $\mathcal{P} = [\![p, q\,; r]\!]$ for some $0 < r < \pi/2$ and $p = p_1/p_2$, $q = q_1/q_2$ rational numbers greater than 2. Combinatorially, $\mathcal{P}$ has Schläfli symbol $\{p_1, q_1\}$. Observe that q_1 is even because the vertex figure is skew antiprismatic. Likewise, p_1 is even. To see this, consider a face f and observe that because the vertex figure is skew the faces adjacent to f lie alternatively on one side and the other of the plane of f. If we travel once around the polygon f we must return to the starting adjacent face and thus p_1 must be even.

The combinatorial dual of $\mathcal{P}$, $\mathcal{P}^*$, can be geometrically realized as $[\![q, p\,; r]\!]$. Indeed, consider the vertex figure $VF(\mathcal{P})$ around a vertex v. The centers of the faces incident to v lie on the symmetry plane of $VF(\mathcal{P})$ and naturally form a polygon of type $[\![q]\!]$. The radius of this polygon is the distance of the center of a face to v, which is precisely r. This defines the vertices, edges and faces of $\mathcal{P}^*$. Finally, observe that its vertex figure is skew of type $[\![p]\!]$.

It is interesting to note that $(\mathcal{P}^*)^* = \mathcal{P}$ geometrically and not only combinatorially as in the classic case in $\mathbb{R}^3$, or in the planar-planar case.

In the rest of this section we give examples of planar-skew polyhedra $[\![p, q\,; r]\!]$ with integer p and q. In the last section we will prove they are the only ones.

5.1 Tori and euclidian planes, $[\![4, 4; r]\!]$

Consider a square $[\![4; r]\!]$. Its internal angle is greater than $\pi/2$, so that to match four of them regularly around a vertex they produce a skew vertex figure of type $[\![4]\!]$. By Lemma 1, we obtain a polyhedron $[\![4, 4; r]\!]$ whose universal cover is the plane tiling $\{4, 4\}$.

Lemma 2 *The polyhedron $\mathcal{P} = [\![4, 4; r]\!]$ is finite if and only if r/π is rational.*

Proof. Consider a vertex v of $\mathcal{P}$, and the four faces around it. Note that the diagonal segment of a face is colinear with the diagonal of the opposite face at v; in fact, they lie on the symmetry plane of the vertex figure. If we follow this line from v we will find vertices of $\mathcal{P}$ after each segment of length $2r$. If r/π is irrational they will be dense and hence infinite.

On the other hand, suppose that $r = (s/k)\pi$. Then these diagonals with vertices of $\mathcal{P}$ form a linear polygon $[\![k/2s]\!]^1$, which has combinatorial length k if k is odd, and $k/2$ if it is even. In the first case, it is the regular polyhedral torus $\{4, 4\}_{k,0}$ and in the second the "slanted" regular polyhedral torus $\{4, 4\}_{k/2,k/2}$ (see, e.g. [5]). □

Observe that the first argument of the preceding proof holds for any planar-skew polyhedron. Let us state this in general:

Corollary 2 *If the planar-skew polyhedron $[\![p, q; r]\!]$ is finite then r/π is rational.* □

Observe also that the only case when the polyhedron $[\![4,4\,;r]\!]$ is an embedded torus is when $r=\pi/2k$, with integer $k>1$. Otherwise the surface it defines has self intersections. See Figure 5.

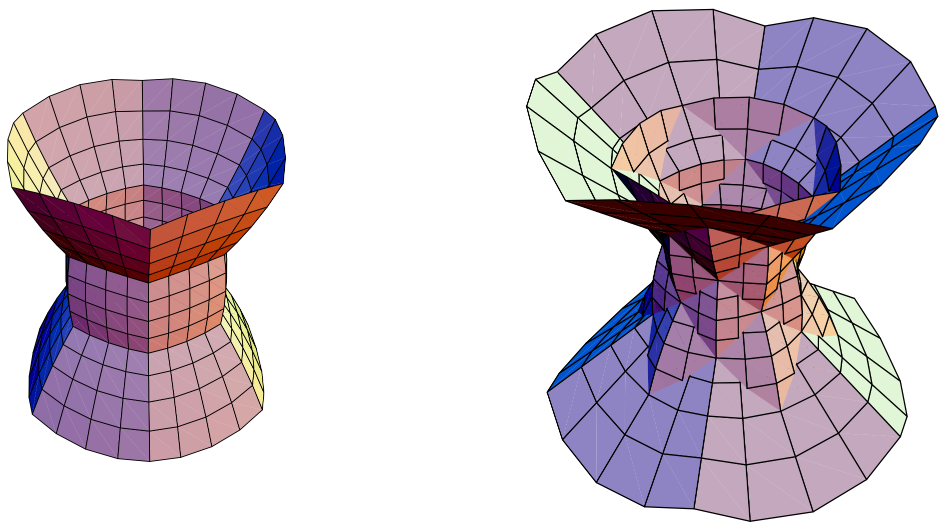

Figure 5: Stereographic projections of $[\![4,4\,;\pi/6]\!]$ and $[\![4,4\,;\pi/5]\!]$. In the first one, the two boundaries are to be antipodally identified. In the second, also the outmost faces are antipodally identified.

The natural geometric way to look at these polyhedra is with their vertices on the quadric surface Q in $\mathbb{P}^3$ defined by the equation $x_1^2+x_2^2=x_3^2+x_4^2$, where $[x_1:x_2:x_3:x_4]$ are homogeneous coordinates. In this quadric, the lines of opposite rulings meet at an angle of $\pi/2$, and generate the tangent plane to Q at their intersection point. Thus, the square $[\![4\,;r]\!]$ centered at this point and with diagonals on the rules has vertices on Q. From this square proceed to build $[\![4,4\,;r]\!]$. The diagonal lines at any given vertex become precisely the two rules of Q. Finally, observe that $[\![4,4\,;r]\!]$ is isometric (and hence equivalent) to its dual.

5.2 The Quinquehedron and the Decahedron

There is a dual pair $[\![4,6\,;\pi/4]\!]$ and $[\![6,4\,;\pi/4]\!]$ with 15 and 10 faces respectively, hence their names. We proceed to describe the Quinquehedron,

$[\![4,6\,;\pi/4]\!]$. Its dual is then constructed in the standard way.

Consider K_5: the complete graph on 5 vertices. Its edge graph has 10 vertices (one for each edge of K_5) and two are adjacent if the corresponding edges are incident, hence it is a regular graph of degree 6. For every cycle of length 4 in K_5, attach the corresponding quadrilateral face to the edge graph. It is easy to see that this is an abstract regular polyhedron, $\mathcal{P}$ say, with automorphism group S_5 (the symmetric group on 5 letters), and Schläfli symbol $\{4,6\}$. Now, to describe its natural embedding in $\mathbb{P}^3$, consider the five basic vectors of R^5 as the vertices of K_5. To each edge of K_5, there corresponds a line, which translated to the origin defines a point in $\mathbb{P}^4$. These ten lines lie in a hyperplane. Thus, the corresponding points lie in a 3-dimensional flat which may be considered as $\mathbb{P}^3$. The edges of $\mathcal{P}$ are taken to be the segments of length $\pi/3$, and then, the radius of each quadrilateral turns out to be $\pi/4$. To see this, consider the same construction with K_3 and K_4. See Figure 6.

Topologically, $[\![4,6\,;\pi/4]\!]$ is a non-orientable surface of genus 7. Its euclidian double cover is the skew $\{4,6\}$ in $\mathbb{R}^4$ discovered by Coxeter in [2].

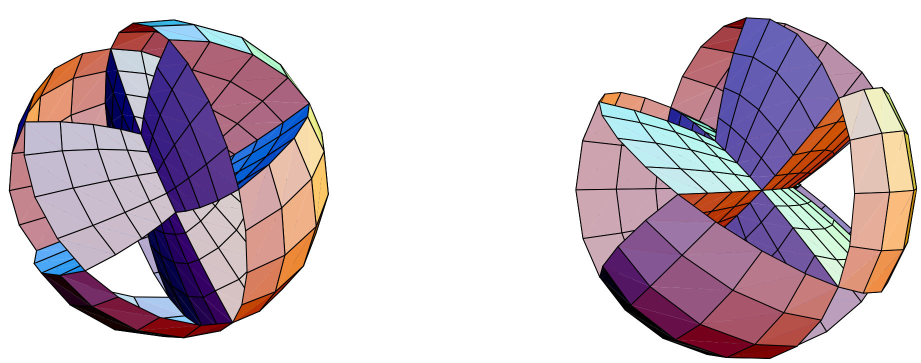

Figure 6: Two views of $[\![4,6\,;\pi/4]\!]$ without one face. The outmost faces are to be identified with their antipodes.

5.3 The Pachyhedron and the Emipachyhedron.

They are the dual pair $[\![4,8\,;\pi/8]\!]$, $[\![8,4\,;\pi/8]\!]$, with 144 and 72 faces respectively. ("Pachy" is the Greek word used for twelve dozens). We follow the description in [11] of $[\![4,8\,;\pi/8]\!]$; see also [12] and [13]. Consider the 12-cell (see [3]), which is a self-dual Polytope consisting of 12 solid octahedra, giving a 3-dimensional tiling or honeycomb of $\mathbb{P}^3$. Shrink each octahedra uniformly and insert triangular prisms with cuadrilateral faces between formerly adjacent pairs of octahedra. (These prisms will be called "waists" bellow). The cuadrilaterals are the faces of $[\![4,8\,;\pi/8]\!]$.

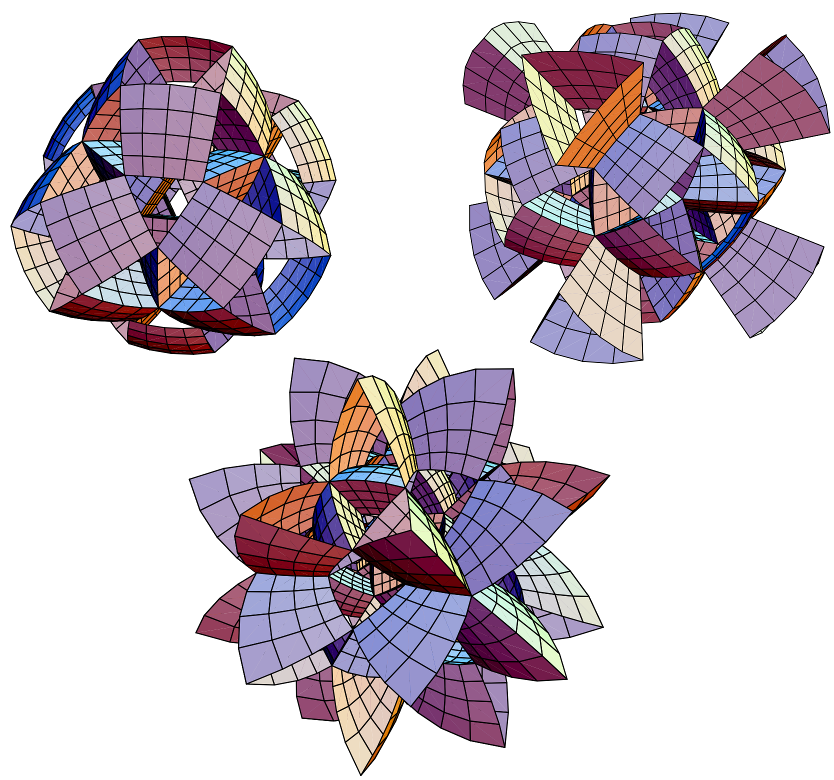

Figure 7: Three stages of $[\![4,8\,;\pi/8]\!]$. In the last two, the outmost faces are to be identified with their antipodes.

6 The waist equation.

Let $\mathcal{P} = [\![p,q\,;r]\!]$ be a finite regular projective polyhedron with planar-inessential face and skew-antiprismatic vertex figure. For the span of this section let us call it a *skew projective polyhedron.*

By a *combinatorial belt* we mean a simple cyclic succession of faces $f_1, f_2, \ldots, f_{c_1}$, such that each face is adjacent to its neighbours through opposite edges (recall that the faces have an even number of sides). The *combinatorial waist* of the polyhedron $\mathcal{P}$ is the length, c_1, of a belt. By regularity, every edge defines a belt and all belts have the same length.

Now we give it a geometric meaning. Consider the plane Π orthogonal to the common edge of f_1 and f_2 at its midpoint. It is orthogonal to the planes of f_1 and f_2 and then, it also intersects the common edge of f_2 and f_3 orthogonally at its midpoint, and so on. The corresponding *geometric belt* is then the planar polygon in Π with vertices at the midpoints of the edges and the segments that contain the centers of the faces. It is therefore of type $[\![c]\!]$ or $[\![c]\!]^{op}$ for some rational $c = c_1/c_2 > 2$. This c is called the *waist* of $\mathcal{P}$.

The geometric belt is inessential if and only if $p_1 \equiv 0 \text{ Mod}(4)$. To see this, consider a planar polygon and an edge on it. This segment defines locally two sides. Observe that if the polygon is inessential the two adjacent edges lie on the same side. And conversely, if it is essential then they lie on opposite sides. Recall that the faces adjacent to a given one, say f, in the polyhedron $\mathcal{P}$ lie on alternating sides of the plane of f as we travel around the polygon f. When we get to the opposite edge we started with, we are on the same side if and only if $p_1 \equiv 0 \text{ Mod}(4)$. This proves the claim on the geometric belt.

Proposition 1 The Waist Equation) *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron with waist c. If $p_1 \equiv 0 \text{ Mod}(4)$, where $p = p_1/p_2$, then*

$$\cos(\pi/p)\cos(\pi/c) = \sin(\pi/q)\cos(r) \ .$$

Proof. Let α be half the internal angle of the face $[\![p\,; r]\!]$, and let β be half the dihedral angle among faces. We claim that

$$\sin(\pi/q) = \sin(\alpha)\sin(\beta) \ . \tag{2}$$

To see this, let b_0, b_1 and b_2 be respectively the barycenters of a distinguished flag. Consider in the tangent space of the vertex, b_0, the infinitesimal euclidian tetrahedron defined by the plane of the distinguished face, the bisecting plane of the dihedral angle at the edge, the symmetry plane of the vertex figure and an orthogonal plane to the edge (infinitesimally close to b_0), (Figure 8). Then from the various right triangles formed, equation (2) follows.

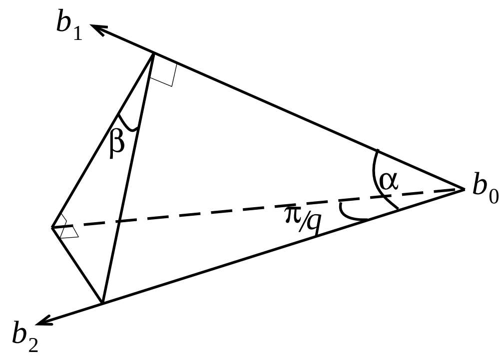

Figure 8: Tetrahedron in the tangent space to b_0.

On the plane of the face, we have a right triangle b_0, b_1, b_2 as in Figure 3, where $\lambda = d(b_0, b_1)$ and $\mu = d(b_1, b_2)$. And equations (1) now become

$$\begin{aligned} \cos(\pi/p) &= \sin(\alpha)\cos(\lambda) \\ \cos(r) &= \cos(\mu)\cos(\lambda) \end{aligned} \tag{3}$$

Finally, consider the geometric belt of $\mathcal{P}$. Since $p_1 \equiv 0$ Mod(4), this polygon is of type $[\![c]\!]$ with side 2μ. Thus the corresponding first equation in (1) yields

$$\cos(\pi/c) = \sin(\beta)\cos(\mu)\ . \tag{4}$$

The proposition follows by expressing $\cos(\pi/p)\cos(\pi/c)$ in terms of equations (3) and (4) and then using (2) and (3). □

Remark. If $p_1 \equiv 2$ Mod(4), similar reasoning leads to the equation $\sin(\pi/2c) = \sin(\pi/q)\sin(r)$. However, we will not use it here.

Proposition 2 *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron such that $q_1 \equiv 2$ Mod(4), where $q = q_1/q_2$. Then $p = 4$ and $r = \pi/4$.*

Proof. Observe that the hypothesis implies that opposite faces at a vertex are coplanar (because opposite vertices of a skew antiprismatic polygon with $q_1 \equiv 2$ Mod(4) sides are colinear with the center of symmetry). Let Π be the plane of a face f. Π contains all opposing faces at the vertices of f and their corresponding ones too, and so on. Draw a point at the barycenter of each face so obtained, and its p_1 radii line segments. Each radius matches with an opposing one to form an edge of a projective graph in Π which is regular (in the strongest geometric sense), and thus is the 1-skeleton of a planar polyhedron. Since p_1 is the number of edges at a vertex of this polyhedron, and it is even, then by Section 2 this polyhedron must be $[\![3,4]\!]$. This proves that $p = 4$. Moreover, $2r$ is the size of the edge of $[\![3,4]\!]$. Thus, $r = \pi/4$. (See Figure 6 for different views of the plane Π and the 3 faces on it.) $\square$

Theorem 1 *Let $\mathcal{P} = [\![p, q\,; r]\!]$ be a skew projective polyhedron with integer p, q and waist c. Then $\mathcal{P}$ is one of the following: $[\![4, 4\,; \pi/k]\!]$ with integer $k \geq 3$, $[\![4, 6\,; \pi/4]\!]$, $[\![6, 4\,; \pi/4]\!]$, $[\![4, 8\,; \pi/8]\!]$ or $[\![8, 4\,; \pi/8]\!]$.*

Proof. If p or q is congruent to $2 \mod(4)$, we may assume by duality that it is q, and then, by Proposition 2, that $p = 4$. Otherwise, p and q are congruent to $0 \mod(4)$. In either case, the waist equation (Proposition 1) holds. From it, since $c \geq 3$ and $\cos r < 1$, we obtain that $\cos \pi/p < 2 \sin \pi/q$. This inequality gives that if $p = 4$ then $q < 9$; that if $p = 8$ then $q < 7$, and that there are no solutions for $p \geq 12$. It is then easy to see that the only integer solutions of the waist equation with our congruence requirements are the following. If $p = q = 4$ then $r = \pi/c$ for any $c \geq 3$. If $p = 4$ and $q = 6$ then $c = 3$ and $r = \pi/4$. If $p = 4$ and $q = 8$ then $c = 3$ and $r = \pi/8$. If $p = 8$ then $q = 4$, $c = 4$ and $r = \pi/8$. The existence of skew projective polyhedra with such invariants was proved in Section 5. $\square$

Observe that the only non-embedded polyhedra of the above list are those in the family $[\![4, 4\,; \pi/k]\!]$ with k odd. However, they lift to the embedded tori $\{4, 4\,|\,k\}$ in $\mathbb{R}^4$ (see [2] and [12]), with the projection being a combinatorial isomorphism. For k even, $[\![4, 4\,; \pi/k]\!]$ is embedded in $\mathbb{P}^3$ and lifts to its double cover $\{4, 4\,|\,k\}$ in $\mathbb{R}^4$. This is so because $-\mathrm{I}$ is a symmetry of $\{4, 4\,|\,k\}$ precisely when k is even. Thus, the lifting to $\mathbb{R}^4$ of the list in the preceding Theorem is Coxeter's one [2].

Finally, observe that if $[\![p, q\,; r]\!]$ is an embedded surface, then its parameters, including the waist, must be integer.

Corollary 3 *The regular projective polyhedra in $\mathbb{P}^3$ with planar face and skew vertex figure that define embedded surfaces are: $[\![4, 4\,; \pi/2k]\!]$ with integer $k \geq 2$, $[\![4, 6\,; \pi/4]\!]$, $[\![6, 4\,; \pi/4]\!]$, $[\![4, 8\,; \pi/8]\!]$ and $[\![8, 4\,; \pi/8]\!]$.*

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