MEAN VALUE FOR THE MATCHING AND DOMINATING POLYNOMIAL

JORGE LUIS AROCHA AND BERNARDO LLANO

ABSTRACT. The mean value of the matching polynomial is computed in the family of all labeled graphs with n vertices. We define the dominating polynomial of a graph whose coefficients enumerate the dominating sets for a graph and study some properties of the polynomial. The mean value of this polynomial is determined in a certain special family of bipartite digraphs.

1. Introduction

The goal of this paper is to compute the average polynomials for the well-known matching polynomial and the dominating polynomial in certain classes of graphs. The matching polynomial first appeared in a paper by Heilman and Lieb [5] as a thermodynamic partition function. For a very interesting introduction to its combinatorial study as well as many of its properties we refer the reader to [2] and [3]. The notion of domination in graphs was introduced last century. This theory can be consulted in the books by Ore [10] and Berge [1]. The paper [7] shows recent developments of the theory and a large account of references on the topic.

In the first part of the paper we calculate the so called average matching polynomial in the class of all labelled graphs with n vertices. The subsequent sections are devoted to the dominating polynomial: definition and basic properties. Finally, we determine the average dominating polynomial in a certain class of bipartite digraphs.

For the terminology of graph theory used here, see [9].

Date. May 29, 1998.

1991 *Mathematics Subject Classification.* Primary 05C70, 05A15.

Key words and phrases. matcching, matching polynomial, dominating set.

This paper is in final form and no version of it will be submitted for publication elsewhere.

2. Average matching polynomial

Consider a simple graph $G=(V,E)$. Let $M\subseteq E$ a matching of the graph G. If M is a matching, then any $M'\subset M$ is a matching too. For $|V|=n$, we have that $|M|\leq n/2$ and if the equality holds, then the matching is called perfect. Let $\alpha_k(G)$ denote the number of matchings of cardinality k ($k\in\mathbb{N}$) of a graph G and by convention, $\alpha_0(G)=1$. The matching polynomial is defined by

$$\alpha(G,t)=\sum_{k=0}^{[n/2]}(-1)^k\alpha_k(G)\,t^{n-2k}.$$

There are basic properties of the matching polynomial studied in [3]. We recall some of these properties that will be used later in this paper.

Theorem 2.1.

$$\alpha(G,t)=\alpha(G-e,t)-\alpha(G-i-j,t),$$

where $i,j\in V$ *and* $e=\{i,j\}\in E$.

Applying this theorem to the complete graph, we have the following result.

Theorem 2.2. *For the complete graph* K_n,

$$\alpha(K_n,t)=He_n(t)=2^{-n/2}H_n\left(t/\sqrt{2}\right)=\sum_{k=0}^{[n/2]}\frac{(-1)^k\,n!}{k!2^k(n-2k)!}t^{n-2k},$$

where

$$\begin{aligned}H_n(t)&=(-1)^n e^{t^2}\frac{d^n}{dx^n}e^{-t^2}\ \text{and}\\ He_n(t)&=(-1)^n e^{t^2/2}\frac{d^n}{dx^n}e^{-t^2/2}\end{aligned}$$

are the Hermite and the special Hermite polynomial respectively.

Further information on Hermite polynomials can be found in [8].

Let $\mathbb{G}_n$ the set of all labeled graphs with n vertices. We define

$$\alpha_n(t)=2^{-\binom{n}{2}}\sum_{G\in\mathbb{G}_n}\alpha(G,t)$$

to be the *mean value of the matching polynomial in the set* $\mathbb{G}_n$ or the *average matching polynomial in* $\mathbb{G}_n$. If $\mathbb{G}_n^k$ denotes the set of all labeled graphs with n vertices and k edges, then we can define the average matching polynomial in this set by

$$\alpha_n^k(t) = \binom{\binom{n}{2}}{k}^{-1} \sum_{G\in\mathbb{G}_n^k} \alpha(G,t). \tag{2.1}$$

It is not difficult to establish that

$$\alpha_n(t) = 2^{-\binom{n}{2}} \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} \alpha_n^k(t). \tag{2.2}$$

Lemma 2.3.

$$\alpha_n^k(t) = \sum_{j=0}^{[n/2]} (-1)^j \binom{k}{j} \binom{\binom{n}{2}}{j}^{-1} \alpha_j(K_n)\, t^{n-2j}.$$

Proof. Applying the definition of the matching polynomial to (2.1), we have that

$$\begin{aligned} \alpha_n^k(t) &= \binom{\binom{n}{2}}{k}^{-1} \sum_{G\in\mathbb{G}_n^k} \sum_{j=0}^{[n/2]} (-1)^j \alpha_j(G)\, t^{n-2j} \\ &= \sum_{j=0}^{[n/2]} (-1)^j \binom{\binom{n}{2}}{k}^{-1} \left[\sum_{G\in\mathbb{G}_n^k} \alpha_j(G)\right] t^{n-2j}. \end{aligned}$$

We compute the sum in brackets. Let $M(G,j)$ be the set of matchings of cardinality j of G, then

$$\sum_{G\in\mathbb{G}_n^k} \alpha_j(G) = \sum_{G\in\mathbb{G}_n^k} \sum_{M\in M(G,j)} 1 \tag{2.3}$$

But any matching of a graph $G \in \mathbb{G}_n^k$ is a matching of the complete graph K_n, so the sum in the right of (2.3) is equal to

$$\sum_{M\in M(K_n,j)} \sum_{\substack{G\in\mathbb{G}_n^k \\ M\in M(G,j)}} 1. \tag{2.4}$$

The second sum of (2.4) is the number of graphs belonging to $\mathbb{G}_n^k$ which contain a fixed matching with exactly j edges. Fixing this matching, from the other $\binom{n}{2} - j$ edges of K_n, we can choose the $k-j$ missing edges of $G \in \mathbb{G}_n^k$ in

$$\binom{\binom{n}{2} - j}{k-j}$$

ways. Therefore, the expression (2.4) is equal to $\binom{\binom{n}{2}-j}{k-j}\alpha_j(K_n)$ and

$$\begin{aligned}\alpha_n^k(t) &= \sum_{j=0}^{[n/2]}(-1)^j\binom{\binom{n}{2}}{k}^{-1}\binom{\binom{n}{2}-j}{k-j}\alpha_j(K_n)\,t^{n-2j}\\ &= \sum_{j=0}^{[n/2]}(-1)^j\binom{k}{j}\binom{\binom{n}{2}}{j}^{-1}\alpha_j(K_n)\,t^{n-2j}\end{aligned}$$

as desired. ∎

Theorem 2.4.

$$\alpha_n(t) = 2^{-n}H_n(t) = \sum_{j=0}^{[n/2]}\left(-\frac{1}{2}\right)^j\alpha_j(K_n)\,t^{n-2j}.$$

Proof. Applying lemma 3.4 to the relation (2.2), we obtain that

$$\begin{aligned}\alpha_n(t) &= 2^{-\binom{n}{2}}\sum_{k=0}^{\binom{n}{2}}\binom{\binom{n}{2}}{k}\sum_{j=0}^{[n/2]}(-1)^j\binom{k}{j}\binom{\binom{n}{2}}{j}^{-1}\alpha_j(K_n)\,t^{n-2j}\\ &= 2^{-\binom{n}{2}}\sum_{j=0}^{[n/2]}(-1)^j\binom{\binom{n}{2}}{j}^{-1}\alpha_j(K_n)\,t^{n-2j}\left[\sum_{k=0}^{\binom{n}{2}}\binom{\binom{n}{2}}{k}\binom{k}{j}\right].\end{aligned}$$

Since

$$\sum_{k=0}^{\binom{n}{2}}\binom{\binom{n}{2}}{k}\binom{k}{j} = 2^{\binom{n}{2}-j}\binom{\binom{n}{2}}{j},$$

then

$$\alpha_n(t) = \sum_{j=0}^{[n/2]}\left(-\frac{1}{2}\right)^j\alpha_j(K_n)\,t^{n-2j}$$

as was to be shown. ∎

3. The dominating polynomial, definition and properties

Let $G=(V,E)$ be a simple graph and $D\subseteq V$. A set of vertices D is said to be a *dominating set* if for every $y\in V-D$, there exists $x\in D$ such that $\{x,y\}\in E$. For any vertex $x\in V$, let $N(x)$ denote the neighbourhood of x, the set of all vertices adjacent to x. We write $N[x]=N(x)\cup\{x\}$, the closed neibourhood of x. With this notation, $D\subseteq V$ is a dominating set if for every $y\in V-D$, we have that $N[y]\cap D\neq\emptyset$. The family of all dominating sets of a graph G is denoted by $\mathcal{D}_G$. Observe that $\emptyset\in\mathcal{D}_G$ and if $D\in\mathcal{D}_G$ and $D\subset D'$, then $D'\in\mathcal{D}_G$.

For any graph G, the number of dominating sets of cardinality k is denoted by $\gamma_k(G)$. We define by

$$\gamma(G,t)=\sum_{i=1}^{n}\gamma_k(G)\,t^{n-k}$$

the *dominating polynomial* of the graph G, where $n=|V|$. Since $\emptyset\in\mathcal{D}_G$, then $\gamma_0(G)=0$.

For example, the dominating polynomials of the complete graph K_n and the totally disconnected graph $\overline{K}_n$ are

$$\gamma(K_n,t)=\sum_{k=1}^{n}\binom{n}{k}t^{n-k}=(1+t)^n-t^n \text{ and } \gamma\left(\overline{K}_n,t\right)=1,$$

since every subset of vertices of K_n is a dominating set and there is only one dominating set of $\overline{K}_n$. If Φ denotes the empty graph, then $\gamma(\Phi,t)=0$ since $\emptyset\in\mathcal{D}_\Phi$.

Let $\bigcup_{i=1}^{n}G_i$ be a graph composed of disjoint subgraphs G_1, G_2, ..., G_n.

Theorem 3.1. $\gamma(G_1\cup G_2,t)=\gamma(G_1,t)\,\gamma(G_2,t)$.

Proof. There are no edges between $V(G_1)$ and $V(G_2)$, therefore $D_1\subseteq V(G_1)$ and $D_2\subseteq V(G_2)$ are dominating sets of G_1 and G_2 respectively if and only if $D_1\cup D_2$ is a dominating set of $G_1\cup G_2$. It holds that $|D_1\cup D_2|=|D_1|+|D_2|$. Then

$$\gamma_k(G_1\cup G_2)=\sum_{i+j=n}\gamma_i(G_1)\,\gamma_j(G_2),$$

which proves the theorem. ∎

As a consequence, we have the following corollary.

Corollary 3.2. $\gamma\left(\bigcup_{i=1}^{n}G_i,t\right)=\gamma(G_1,t)\,\gamma(G_2,t)\,...\gamma(G_n,t)$.

Let G_1+G_2 be the sum of graphs $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ defined as $G_1+G_2=(V_1\cup V_2,E)$, where

$$E=E_1\cup E_2\cup\{\{x,y\}:x\in V_1,y\in V_2\}.$$

Theorem 3.3. *Let $G_1=(V_1,E_1)$ and $G_2=(V_2,E_2)$ be any graphs such that $|V_1|=n_1$ and $|V_2|=n_2$. Then*

$$\begin{aligned}\gamma(G_1+G_2,t)=\gamma(K_{n_1+n_2},t)-t^{n_1}\left[\gamma(K_{n_2},t)-\gamma(G_2,t)\right]\\ -t^{n_2}\left[\gamma(K_{n_1},t)-\gamma(G_1,t)\right].\end{aligned}$$

Proof. Let D be a dominating set of G_1+G_2 such that $|D|=k$. We define the following sets:

$$\begin{aligned} S_{1,2} &= \{D \subseteq V_1 \cup V_2 : D \text{ is a dominating set in } G_1 \text{ and } G_2\}, \\ S_{\overline{1},2} &= \{D \subseteq V_1 : D \text{ is not a dominating set in } G_1\} \text{ and} \\ S_{1,\overline{2}} &= \{D \subseteq V_2 : D \text{ is not a dominating set in } G_2\}. \end{aligned}$$

With this notation, we have that

$$|S_{1,2}| = \binom{n_1+n_2}{k} - \left|S_{\overline{1},2}\right| - \left|S_{1,\overline{2}}\right|.$$

Therefore

$$\gamma_k(G_1+G_2) = \binom{n_1+n_2}{k} - \left[\binom{n_1}{k} - \gamma_k(G_1)\right] - \left[\binom{n_2}{k} - \gamma_k(G_2)\right].$$

Multiplying by t^{n-k} and summing for all $k=1,...,n$, the desired result is established. ∎

This theorem can be applied to compute the dominating polynomial of the complete n-partite graph $K_{m_1,m_2,\ldots,m_n} = \overline{K}_{m_1} + \overline{K}_{m_2} + \ldots + \overline{K}_{m_n}$:

$$\gamma(K_{m_1,m_2,\ldots,m_n}, t) = (1+t)^m - t^m - \sum_{i=1}^{n} t^{m-m_i}\left[(1+t)^{m_i} - t^{m_i} - 1\right],$$

where $m=\sum_{i=1}^n m_i$.

Consider now a digraph $\Gamma=(U,A)$, where U and A denote the set of vertices and arcs respectively. The sets of the ex-neighbourhood and in-neighbourhood of a vertex x are denoted by $N^+(x)$ and $N^-(x)$ respectively and write $N^+[x]$ and $N^-[x]$ for the respective closed neighbourhoods. We say that $D\subseteq U$ is a dominating set of Γ if for every vertex $v\in U-D$, there exists $u\in D$ such that $(u,v)\in A$, that is, $N^-[v]\cap D\neq\emptyset$ The dominating polynomial of a digraph Γ is defined similarly as for graphs. The properties proved before are valid in this case too.

Let us call a bipartite digraph $\Gamma=(U_1,U_2,A)$ *one-way* if its arcs are all directed from part U_1 to part U_2. The family of dominating sets of the one-way bipartite digraph Γ is denoted by $\mathcal{D}_\Gamma$. With this definition, $\mathcal{D}_\Gamma\subseteq 2^{U_1}$. Observe that if $U_1=\emptyset$, then $\gamma(\Gamma,t)=0$ (there is no dominating set) and if $U_2=\emptyset$, then $\gamma(\Gamma,t)=(1+t)^n$ by convention.

Let $G=(V,E)$ be a graph. We construct a one-way bipartite digraph $\widetilde{G}=(U_1,U_2,A)$ from the graph G such that U_1 and U_2 are disjoint copies of the set V and

$$A=\{(i,i) : i\in V\}\cup\{(i,j),(j,i) : \{i,j\}\in E\}.$$

Lemma 3.4. $\gamma(G,t) = \gamma\left(\widetilde{G}, t\right)$.

Proof. It is enough to show that $\mathcal{D}_G = \mathcal{D}_{\widetilde{G}}$. The relation $\mathcal{D}_G \subseteq \mathcal{D}_{\widetilde{G}}$ is evident. Conversely, suppose that there exists $D \in \mathcal{D}_{\widetilde{G}}$ such that $D \notin \mathcal{D}_G$. Then for every $y \notin D$ we have that $N^-_{\widetilde{G}}[y] \cap D \neq \emptyset$ and there exists $y \notin D$ such that $N[y] \cap D = \emptyset$. But there exists $x \in D$ such that $(x, y) \in A$ and by the construction of $\widetilde{G}$, then $\{x, y\} \in E$, which is a contradiction. ∎

Theorem 3.5. *For any one-way bipartite digraph* $\Gamma = (U_1, U_2, A)$ *and* $i \in U_1$,

$$\gamma(\Gamma, t) = t\gamma(\Gamma - i, t) + \gamma\left(\Gamma - N^+[i], t\right).$$

Proof. The number of dominating sets of cardinality k in Γ splits into two parts:

(i) The number of dominating sets of cardinality k not containing vertex i, i.e. the number of dominating sets of cardinality k in $\Gamma - i = (U_1 - i, U_2, A - A')$, where

$$A' = \{(i,i)\} \cup \{(i,j) : j \in U_2\}.$$

(ii) The number of dominating sets containing vertex i. Let $i \in D$, where D is a dominating set and $|D| = k$. Delete vertex i and all the vertices dominated by it. Then the number of dominating sets of cardinality k is equal to the number of those sets, but of cardinality $k-1$ in $\Gamma - N^+[i] = (U_1 - i, U_2 - N^+(i), A - A'')$, where

$$A'' = A' \cup \left\{(x,y) : x \in N^-(y), y \in N^+(i)\right\}$$

These sets can be chosen in $\gamma_{k-1}(\Gamma - N^+[i])$ ways. Therefore

$$\gamma_k(\Gamma) = \gamma_k(\Gamma - i) + \gamma_{k-1}\left(\Gamma - N^+[i]\right).$$

Multiplying this equality by t^{n-k} and summing for all $k = 1, ..., n$, we obtain the result. ∎

Observe that lemma 3.4 and theorem 3.5 imply that the recurrence

$$\gamma(G,t) = t\gamma\left(\widetilde{G} - i, t\right) + \gamma\left(\widetilde{G} - N^+[i], t\right)$$

holds for any graph G.

4. Average dominating polynomial

Let us consider the dominating polynomial $\gamma(\Gamma, t)$ of an one-way bipartite digraph Γ as a random variable, whose average value in the family $\mathbb{D}_{n,m}$ of all labeled bipartite graphs with partite sets of size n and m respectively, is define by

$$\gamma_{n,m}(t) = \frac{1}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma(\Gamma, t). \tag{4.1}$$

This polynomial is called the *average dominating polynomial of the family* $\mathbb{D}_{n,m}$.

Theorem 4.1.

$$\gamma_{n,m}(t) = \sum_{k=0}^{n-1} \binom{n}{k} \left(1 + \frac{1}{2^{n-k}}\right)^m t^k, \; n, m \geq 1.$$

Proof. Let $\Gamma = (U_1, U_2, A) \in \mathbb{D}_{n,m}$. Applying theorem 3.5 to (4.1), we obtain that

(4.2)
$$\gamma_{n,m}(t) = \frac{t}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma(\Gamma - i, t) + \frac{1}{2^{nm}} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma\left(\Gamma - N^+[i], t\right).$$

Observe that

$$\sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma(\Gamma - i, t) = 2^m \sum_{\Gamma \in \mathbb{D}_{n-1,m}} \gamma(\Gamma, t),$$

since vertex i can be joined to each one-way bipartite digraph of $\mathbb{D}_{n-1,m}$ in 2^m ways. On the other hand, if $|N^+(i)| = k$, then the second sum in the right of (4.2) runs through all labeled one-way bipartite digraphs of the family $\mathbb{D}_{n-1,m-k}$. The labels of the k vertices of $N^+(i) \subseteq U_2$ can be chosen in $\binom{m}{k}$ ways, there are no edges between them and these k vertices can be joined to the $n-1$ vertices of the set U_1 in $2^{(n-1)k}$ ways. Then

$$\begin{aligned} \sum_{\Gamma \in \mathbb{D}_{n,m}} \gamma(\Gamma - N[i], t) &= \sum_{k=0}^{m} \binom{m}{k} 2^{(n-1)k} \sum_{\Gamma \in \mathbb{D}_{n-1,m-k}} \gamma(\Gamma, t) \\ &= \sum_{j=0}^{m} \binom{m}{j} 2^{(n-1)(m-j)} \sum_{\Gamma \in \mathbb{D}_{n-1,j}} \gamma(\Gamma, t). \end{aligned}$$

Therefore

$$\begin{aligned}\gamma_{n,m}(t) &= \frac{t}{2^{(n-1)m}} \sum_{\Gamma\in\mathbb{D}_{n-1,m}} \gamma(\Gamma,t) \\ &\quad + \frac{1}{2^{nm}} \sum_{j=0}^{m} \binom{m}{j} 2^{(n-1)(m-j)} \sum_{\Gamma\in\mathbb{D}_{n-1,j}} \gamma(\Gamma,t)\end{aligned}$$

and so

$$\gamma_{n,m}(t) = t\gamma_{n-1,m}(t) + \frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}\gamma_{n-1,j}(t). \tag{4.3}$$

Let us consider the following exponential generating function:

$$\gamma(x,y,t) = \sum_{n\geq 0}\sum_{m\geq 0}\gamma_{n,m}(t)\frac{x^n}{n!}\frac{y^m}{m!}.$$

Multiplying (4.3) by $x^n y^m/n!m!$ and summing for all $n, m \geq 0$, we have that

$$\begin{aligned}\sum_{n\geq 0}\sum_{m\geq 0}\gamma_{n,m}(t)\frac{x^n}{n!}\frac{y^m}{m!} &= t\sum_{n\geq 0}\sum_{m\geq 0}\gamma_{n-1,m}(t)\frac{x^n}{n!}\frac{y^m}{m!} \\ &\quad + \sum_{n\geq 0}\sum_{m\geq 0}\frac{1}{2^m}\frac{x^n}{n!}\frac{y^m}{m!}\sum_{j=0}^{m}\binom{m}{j}\gamma_{n-1,j}(t).\end{aligned}$$

From this formula,

$$\frac{\partial\gamma(x,y,t)}{\partial x} = t\gamma(x,y,t) + e^{\frac{y}{2}}\gamma\left(x,\frac{y}{2},t\right), \tag{4.4}$$

since

$$\begin{aligned}&\sum_{n\geq 0}\sum_{m\geq 0}\frac{x^n}{n!}\frac{\left(\frac{y}{2}\right)^m}{m!}\sum_{j=0}^{m}\binom{m}{j}\gamma_{n-1,j}(t) \\ &= \sum_{n\geq 0}\sum_{j\geq 0}\frac{x^n}{n!}\gamma_{n-1,j}(t)\sum_{j=0}^{m}\binom{m}{j}\frac{\left(\frac{y}{2}\right)^m}{m!} \\ &= \sum_{n\geq 0}\sum_{j\geq 0}\frac{x^n}{n!}\gamma_{n-1,j}(t)\frac{\left(\frac{y}{2}\right)^j}{j!}\sum_{m\geq 0}\frac{\left(\frac{y}{2}\right)^{m-j}}{(m-j)!} \\ &= e^{\frac{y}{2}}\gamma\left(x,\frac{y}{2},t\right).\end{aligned}$$

Let us find the solution of the partial differential equation (4.4) in the following form

$$\gamma(x,y,t) = e^y f(x,y,t). \tag{4.5}$$

Consequently,

$$\frac{\partial f(x,y,t)}{\partial x} = tf(x,y,t) + f\left(x, \frac{y}{2}, t\right).$$

If

$$f(x,y,t) = \sum_{n\geq 0}\sum_{m\geq 0} f_{n,m}(t)\frac{x^n}{n!}\frac{y^m}{m!},$$

then

$$f_{n,m}(t) = \left(t + \frac{1}{2^m}\right) f_{n-1,m}(t) = \left(t + \frac{1}{2^m}\right)^{n-1} f_{1,m}(t).$$

Computing $f_{1,m}(t)$, we have from (4.5) that

$$\gamma_{1,m}(t) = \sum_{j=0}^{m} \binom{m}{j} f_{1,m}(t).$$

Since $\gamma_{1,m}(t) = 1/2^m$ for $m \geq 1$ and $\gamma_{1,0}(t) = 1 + t$, then

$$f_{1,m}(t) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \gamma_{1,m}(t) = (-1)^m \left(t + \frac{1}{2^m}\right).$$

Therefore

$$f_{n,m}(t) = (-1)^m \left(t + \frac{1}{2^m}\right)^n$$

and

$$\gamma_{n,m}(t) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left(t + \frac{1}{2^j}\right)^n = \sum_{k=0}^{n-1} \binom{n}{k} \left(1 + \frac{1}{2^{n-k}}\right)^m t^k.$$

∎

Observe that the recurrence for the dominating polynomial is closed in the family $\mathbb{D}_{n,m}$. The problems of finding a recurrence relation for the dominating polynomial which is closed in the family $\mathbb{G}_n$, and the calculation of the average polynomial in this family remain open. The same questions can be posed for polynomials defined for other invariants of graphs, such as minimal dominating sets, K_n-dominating sets (see [7]), vertex-coverings and edge-coverings.

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, London, 1973).

[2] E.J. Farrell, An introduction to matching polynomials, J. Combinatorial Theory Ser B 27 (1979) 75-86.

[3] C.D. Godsil and I. Gutman, On the theory of the matching polynomial, J. Graph Theory 5 (1981) 137-144.

[4] C.D. Godsil, Algebraic Combinatorics (Chapman and Hall, New York, 1993).

[5] O.J. Heilmann and E.H. Lieb, Monomers and dimers, Phys. Rev. Lett. 24 (1970) 1412-1414.
[6] O.J. Heilmann and E.H. Lieb, Theory of monomer-dimer systems, Comm. Math. Phys. 25 (1972) 190-232.
[7] M.A. Henning, O.R. Oellermann and H.C. Swart, The diversity of domination, Discrete Math. 161 (1996) 161-173.
[8] N.N. Lebedev, Special Functions and their Applications (Dover, New York, 1972).
[9] L. Lovász, Combinatorial Problems and Exercises (North-Holland, Amsterdam, 1979).
[10] O. Ore, Theory of Graphs (Amer. Math. Soc., Providence, 1962).

Instituto de Matemáticas, UNAM, Circuito Exterior, Ciudad Universitaria, México, D.F. 04510
E-mail address: arocha@math.unam.mx

E-mail address: bllano@math.unam.mx