The Size of Minimum 3-Trees: Case $2 \bmod 3$

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ABSTRACT. A 3-uniform hypergraph is called a minimum 3-tree, if for any 3-coloring of its vertex set there is a heterochromatic triple and the hypergraph has the minimum possible number of triples. There is a conjecture that the number of triples in such 3-tree is $\left\lceil \frac{n(n-2)}{3} \right\rceil$ for any number of vertices n. In [4] Sterboul gave a proof that this is true when $n \equiv 2 \bmod 3$, however his proof is incomplete. Here we give a full proof of this case using the basic construction and the main ideas of [4] .

Keywords: Tight hypergraphs, triple systems.
Mathematics Subject Classification (1991). 05B07, 05C65, 05D10.

1. INTRODUCTION

A 3-*graph* is an ordered pair of sets $G = (V, \Delta)$. The elements of V are called *vertices*. The elements of Δ are subsets of vertices of cardinality 3 and are called *triples*. Given a 3-graph $G = (V, \Delta)$ and a vertex v the trace $Tr_G(v)$ of v in G is the graph with vertex set $V \setminus \{v\}$, and a pair $\{x, y\}$ is an edge of $Tr_G(v)$ if and only if $\{v, x, y\}$ is a triple of G. Henceforth, the number of vertices in a 3-graph will be denoted by n.

A 3-graph is called *tight* (see [1]) if any proper 3-partition (3-coloring) of the vertex set has a transversal (heterochromatic) triple. A tight 3-graph is called a 3-*tree*, if whenever we delete a triple from it we obtain an untight 3-graph. Different 3-trees on n vertices may have a different number of triples. From the results of [3],

*Partially supported by CONACYT grant No: 400333-5-27968E

we know that the maximum number of triples in any 3-tree is $\binom{n-1}{2}$. It is not difficult to show that the minimum number of triples in such a 3-tree is not less than $\left\lceil \frac{n(n-2)}{3} \right\rceil$. In [1] it was proved that this bound is sharp for any n of the form $\frac{p-1}{2}$ where p is a prime number, and it was conjectured that the bound is sharp for any n. In [2] the case when $n \equiv 3, 4 \bmod 6$ was solved.

In [4] Sterboul gave a proof that this is true when $n \equiv 2 \bmod 3$, however his proof is incomplete and lacks many details. Here we give a full proof of this case using the basic construction and the main ideas of [4] .

2. THE CONSTRUCTION

The remark in the introduction shows that to prove the conjecture for any n it is sufficient to construct a 3-tree with $\left\lceil \frac{n(n-2)}{3} \right\rceil$ triples. Here we are dealing only with the case $n \equiv 2 \bmod 3$ which has sense when $n \geq 5$. For $n = 5$ the construction in [1] gives the result.

So, let $n = 3t + 2$, $t \geq 2$. Let us consider the cyclic group $\mathbb{Z}_n = \{0, 1, ..., n-1\}$, its elements are the vertices of the 3-graphs.

Of course, we know how to add vertices. If $e = \{x_1, x_2, x_3\}$ is a triple and y is a vertex, then $e + y = \{x_1 + y, x_2 + y, x_3 + y\}$. If F is any set of triples and S any set of vertices then $F + S = \{f + s | f \in F, s \in S\}$ .It is important to observe that all sum operations are $\bmod n$.

An important element of $\mathbb{Z}_n$ in the construction is the number $l = \left\lceil \frac{t-1}{2} \right\rceil$ where the operations involved in its definition are performed in $\mathbb{Q}$.

Let us denote by

$$\mathbb{B}_n = \{-4, 4, -7, 7, ..., -(3l+1), 3l+1\} \subset \mathbb{Z}_n$$

and observe the following properties of $\mathbb{B}_n$:

- $x \in \mathbb{B}_n \setminus \{4\} \Rightarrow x - 3 \in \mathbb{B}_n$.
- $x \in \mathbb{B}_n \setminus \{-4\} \Rightarrow x + 3 \in \mathbb{B}_n$.
- $\{\{-2, -1, 0, 1, 2\}, \mathbb{B}_n - 1, \mathbb{B}_n, \mathbb{B}_n + 1\}$ is a partition of $\mathbb{Z}_n$.

For $x \in \mathbb{B}_n$, let us consider the following triples:

$$\begin{aligned} \alpha &= \{0, 1, 2\} \\ \beta_x &= \{0, 1, x\} \end{aligned}$$

Those triples generate the set of triples in the 3-graph i.e. any triple will be of the form $\alpha + y$ or $\beta_x + y$ where $y \in \mathbb{Z}_n$. Formally, denote $G_n = (\mathbb{Z}_n, (\alpha \cup \{\beta_x | \ x \in \mathbb{B}_n\}) + \mathbb{Z}_n)$. Our purpose is to show that G_n is a 3-tree with $\frac{n(n-2)}{3}$ triples.

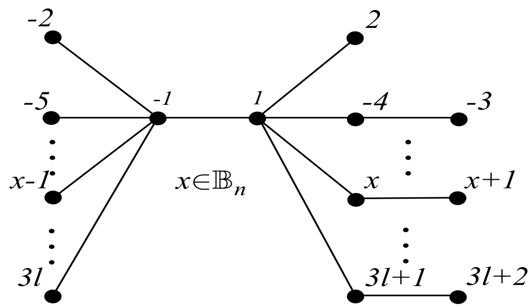

Figure 1: The trace of the vertex 0

Let us consider the trace of the vertex 0. It is defined by all triples containing 0. Actually, those triples are $\alpha, \alpha - 1$, $\alpha - 2$, β_x, $\beta_x - 1$, and $\beta_x - x$ for $x \in \mathbb{B}_n$. So, the trace of cero is the graph (in fact, it is a tree) represented in the figure.

Let $y \in \mathbb{Z}_n$ and consider the function $g_y : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ defined by $g_y\left(x\right) = x + y$ for any vertex x of G_n. Clearly g_y is an automorphism of G_n, hence the trace of y is a graph isomorphic to the trace of 0. Therefore, the trace of any vertex in G_n is isomorphic to the same tree. By a standard counting argument we obtain that G_n has $\frac{n(n-2)}{3}$ triples. So, G_n has the required number of triples and if we prove that it is tight, then the proof of the conjecture follows for our case.

Theorem 1. *G_n is tight .*

Proof. Let $f : \mathbb{Z}_n \to \{R, B, Y\}$ be a proper red-blue-yellow coloring of $\mathbb{Z}_n$ having no transversal triple in G_n. Since the trace of any vertex is connected, then any color has at least two vertices in its preimage by f.

A repeating argument in the proof below is the following. Suppose we have two triples $t_1 = \{x_1, y_1, z\}$ and $t_2 = \{x_2, y_2, z\}$ of the 3-graph G_n having the vertex z in common and that we know the colors of x_1, y_1, x_2 and y_2. Suppose moreover that $f(x_i) \neq f(y_i)$ $i \in \{1, 2\}$ and $\{R, B, Y\} = \{f(x_i), f(y_i)|\ i \in \{1, 2\}\}$ hold. Under those conditions and the assumption that f is non-heterochromatic we would know the color of z. Arguments of this kind will be used repeatedly and we introduce the following notation for it. Let, for example $f(x_1) = R$, $f(y_1) = B$, $f(x_2) = B$ and $f(y_2) = Y$ be the colors of those vertices, then we know that $f(z) = B$ and we will write:

$$\left\{ \begin{array}{c} t_1 = \{x_1, y_1, z\}, t_2 = \{x_2, y_2, z\} \in G_n \\ f(x_1) = R,\ f(y_1) = B,\ f(x_2) = B,\ f(y_2) = Y \end{array} \right\} \Rightarrow f(z) = B$$

We shall divide the proof into two cases, namely, the first case is when $f\left(0\right) \neq f\left(-1\right) = f\left(1\right)$ and the second one is when this is not true.

Let us settle the first case and assume that $f(0) = R$ and $f(1) = f(-1) = B$. Since the coloring is proper there must be a vertex x such that $f(x) = Y$. A quick glance at the trace of 0 in the figure and the assumption that f is non-heterochromatic shows to us that $x \in \mathbb{B}_n + 1$ and we can suppose that $x \neq -3$ (since there are at least two yellow vertices). Observe that in this case $\{x-1, -(x-1), x+2\} \subset \mathbb{B}_n$. Therefore, we have

$$\left\{ \begin{array}{c} \beta_{x-1} = \{0, 1, x-1\}, \beta_{-(x-1)} + x - 1 = \{0, x, x-1\} \in G_n \\ f(0) = R,\ f(1) = B,\ f(x) = Y \end{array} \right\} \Rightarrow f(x-1) = R$$

and

$$\left\{ \begin{array}{c} \beta_{x+2} - 1 = \{-1, 0, x+1\}, \alpha + (x-1) = \{x-1, x, x+1\} \in G_n \\ f(-1) = B,\ f(0) = R,\ f(x-1) = R,\ f(x) = Y \end{array} \right\} \Rightarrow f(x{+}1) = R.$$

On the other hand, the triple $\beta_{-(x-1)} + x = \{1, x, x+1\}$ is heterochromatic. This is a contradiction which shows that the first case is not possible.

Let us consider the second case. If there exists $x \in \mathbb{Z}_n$ such that $f(x) \neq f(x-1) = f(x+1)$ then the coloring g defined by $g(z) = f(z-x)$ for $z \in \mathbb{Z}_n$ is also non-heterochromatic, $g(0) \neq g(-1) = g(1)$ and the first case applies giving a contradiction. So we know that if $f(x) \neq f(x-1)$ then $f(x-1) \neq f(x+1)$. Moreover, since $\alpha + x - 1 = \{x-1, x, x+1\} \in G_n$ is non-heterochromatic we have that $f(x) = f(x+1)$.

This fact $(f(x) \neq f(x-1) \Rightarrow\ f(x) = f(x+1))$ used twice, yields the following property that holds for the rest of the proof:

$$f(x) \neq f(x-1) \Longrightarrow (f(x-2) = f(x-1) \text{ and } f(x) = f(x+1)) \qquad (*)$$

Without lost of generality we may suppose that $f(0) = R$ and $f(1) = B$ and by (*) we have $f(2) = B$ and $f(-1) = R$.

Claim 1. *For all $x \in \mathbb{B}_n - 1$, if $f(x) = Y$ then $f(x+1) = R$ and $f(x-1) = Y$.*

Proof of the Claim . Let $x \in \mathbb{B}_n - 1$ and suppose that $f(x) = Y$ then we have

$$\left\{ \begin{array}{c} \beta_{x+1} = \{0, 1, x+1\}, \beta_{-(x+1)} + x = \{-1, x, x+1\} \in G_n \\ f(-1) = R,\ f(0) = R,\ f(1) = B,\ f(x) = Y \end{array} \right\} \Rightarrow f(x+1) = R.$$

by (*) $f(x-1) = Y$ and the proof of the claim is completed.

Claim 2. *For all $x \in \mathbb{B}_n + 1$, if $f(x) = Y$ then $f(x-1) = R$ and $f(x+1) = Y$.*

Proof of the Claim . Let $x \in \mathbb{B}_n + 1$ and suppose that $f(x) = Y$ then we have

$$\left\{ \begin{array}{c} \beta_{x-1} = \{0, 1, x-1\}, \beta_{-(x-1)} + x - 1 = \{0, x, x-1\} \in G_n \\ f(0) = R,\ f(1) = B,\ f(x) = Y \end{array} \right\} \Rightarrow f(x-1) = R$$

and by (*) $f(x+1) = Y$ and the proof of the claim is completed.

Let x be a yellow vertex. If $x = -2$ then the triple $\beta_4 - 2 = \{-2, -1, 2\}$ gives a contradiction. Observing the trace of 0 we notice that $x \in \{\mathbb{B}_n - 1, \mathbb{B}_n + 1\}$. If $x = -3$, then -2 is yellow by Claim 2 and if $x = 3$, then 2 is yellow by Claim 1 which are contradictions. Therefore, by claims 1 and 2 we can suppose that $x \in \mathbb{B}_n + 1$ and $x + 1 \in \mathbb{B}_n - 1$ are yellow vertices. On the other hand, 2 is blue, $x + 1$ is yellow and by the Claim 1, $x + 2$ is red and the triple $\beta_{-(x-1)} + (x + 1) = \{2, x + 1, x + 2\}$ is heterochromatic. This concludes the proof of the Theorem.

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