Long Induced Paths in 3-connected Planar Graphs.

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Abstract

It is shown that every 3-connected planar graph with a large number of vertices has a long induced path.

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Let G be an undirected graph without loops and multiple edges. Denote by $p(G)$ the number of vertices in the longest induced path of G . Finding long induced paths in graphs is an interesting but difficult problem. However, it is easy to revise all the references devoted to related problems (see [1-7]).

Denote $p_n = \min\{p(G)\}$ where the minimum is taken over all triconnected planar graphs of order n. The purpose of this note is to prove the following.

Theorem. $\lim_{n \to \infty} p_n = \infty$

Proof. Denote by G_n a fixed triconnected planar graph such that $p(G_n) = p_n$. Let Δ_n be the maximum degree of G_n and let v_n be a fixed vertex of maximum degree in G_n . It is easy to see that the diameter d of any graph is large if it has an small maximum degree. In fact one can prove that $p_n \geq d(G_n) + 1 \geq \log_{\Delta_n} n$. So if $\{\Delta_n\}$ is bounded, then we are done. Hence, we can suppose that $\{\Delta_n\}$ grows.

A well known theorem of Whitney states that, any triconnected planar graph has an unique embedding in the sphere. In this embedding the topological neighborhood of a vertex v is an open disk bounded by a cycle C_v of the graph which in general contains more vertices than the ones in the graphical neighborhood of the vertex.

Denote by G'_n the graph obtained from G_n by deleting v_n and every other vertex not in C_{v_n}. Of course, any induced path in G'_n is an induced path in G_n. We denote by n' the order of G'_n. We know that $n' \geq \Delta_n$ and therefore $\{n'\}$ is unbounded.

We can think on the graph G'_n as drawn in the plane in such a way that the cycle C_{v_n} bounds the infinite face. Let D_n be the dual graph of G'_n and let us delete from D_n the vertex corresponding to the infinite face to obtain D'_n. Since every vertex of G'_n lies in the boundary of the infinite face then, D'_n is a tree.

Let us associate to each vertex of D'_n a weight equal to the number of vertices of the corresponding face in G'_n minus two. The weight of a path in D'_n is by definition the sum of the weights of its vertices. Observe that a path of weight w in D'_n corresponds to a subgraph P of G'_n which is a path of faces separated by edges. It is easy to see that P has exactly $w + 2$ vertices. Deleting a vertex from each of the two end faces of P we split the boundary of P into two paths. Again, the fact that every vertex of G'_n lies in the boundary of the infinite face implies that these two paths are induced in G'_n and one of them has at least $w/2$ vertices. Therefore, if we denote by w_n the maximum weight of a path in D'_n then, to prove the proposition we must show that $\{w_n\}$ is unbounded.

Denote by $k = k(n)$ the size of the biggest interior face in G'_n and by $m = m(n)$ the number of vertices in D'_n. If we triangulate all interior faces of G'_n , then the number of all interior triangles with respect to the cycle C_{v_n} must be

$n' - 2$, but in the interior of each face there are at most $k-2$ triangles and so $m \geq \frac{n'-2}{k-2}$. Let v be a vertex in D'_n of eccentricity equal to the diameter $d = d(n)$ of D'_n and denote by V_i the set of vertices at distance i from v.

It is clear that

$$\frac{n'-2}{k-2} \leq m = \sum_{i=0}^{d} |V_i| \leq \sum_{i=0}^{d} k^i \leq \frac{k^{d+1}-2}{k-2}$$

and therefore $\log_3 n' \leq (d+1)\log_3 k$. Since any vertex has weight no less than one then $w_n \geq d+1$. On the other hand, $w_n \geq k-2 \geq \log_3 k$ for any $k \geq 3$. Hence, $w_n \geq \sqrt{\log_3 n'}$ and the proof is completed. $\square$

Remark. The method in the proof of the proposition gives a lower bound $O(\log n)$ for maximal outerplanar graphs with n vertices. However, this an easier result that can be proved in several other ways. In this case the bound is asymptotically sharp. It is reached in the family $\{\mathbf{S}_i\}$ shown in the figure.

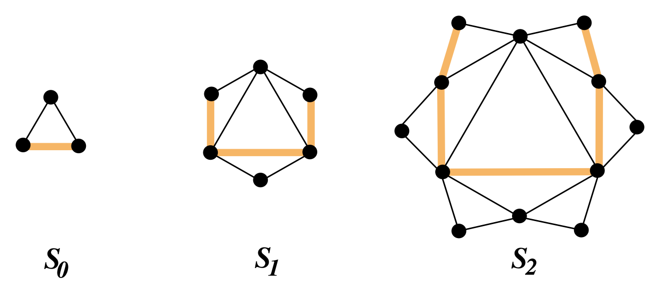

Figure 1: Polygon triangulations with $p = O(\log n)$

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