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CLASSIFICATION OF NON-DEGENERATE QUASIHOMOGENEOUS FUNCTIONS WITH INNER MODALITY 6

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1. INTRODUCCION.

The analysis of the normal forms to which functions can be reduced in neighbourhoods of degenerate critical points (V. I. Arnold, [1] and [2]) shows that many of them are simple and correspond to quasihomogeneous or semiquasihomogeneous functions (a semiquasihomogeneous function is a sum of a quasihomogeneous function with an isolated critical point and summands of higher generalizaed degree).

In this paper we continue the classification of non-degenerate quasihomogeneous functions, which was begun by V.A. Arnold for inner modality 0 and 1, and further developed by E. Yoshinaga and M. Suzuki and the authors for inner modality 2, 3, 4 and 5 ([1], [2], [5], [6] and [12]). In paragraph 3 we state the main theorems which are tables of all normal forms of non-degenerate quasihomogeneous polynomials with inner modality 6 and all normal forms of non-degenerate quasihomogeneous polynomials which are boundaries of non-degenerate quasihomogeneous polynomials with inner modality 6.

In paragraph 5 we study some of the adjacentnesses among these new classes of quasihomogeneous polynomials and extend to $k \leqslant 5$ the following proposition: *Each family of non-degenerate quasihomogeneous polynomials with inner modality* $k+1$ *is adjacent to one of the families of boundaries of non-degenerate quasihomogeneous polynomials with inner modality* k.

The group of germs (or jets) of diffeomorphisms of C^n, leaving 0 invariant, acts on the space of germs (or jets) of functios at 0. A *singularity class* or *family* is a subset of the space of functions invariant under this action.

Two germs (or jets) belonging to the same orbit are called *equivalent*; they are called *stably equivalent* if they become equivalent by addition of a non-degenerate quadratic form.

An important result of J. Mather (see [8]) states that every smooth function with an isolated critical point at 0 is equivalent to a fairly long segment of its Taylor series. Thus, in investigating isolated critical points, the functions may be replaced by polynomials and the question of classifying critical points reduces to an algebraic question about the orbits of the action of a finite-dimensional Lie group on a finite-dimensional manifold.

The *number of moduli* (or the *modality*) of a function in the neighbourhood of a critical point is defined as follows.

Let G be a Lie group acting on a finite-dimensional manifold X. The orbits of G can generate discrete stratifications or continuous families in the neighbourhood of a point $x \in X$. We say that a point x has *modality* m (under the given action) if a sufficiently small neighbourhood of x in X can be covered by finitely many families of orbits, depending on no more than m parameters (and an arbitrarily small neighbourhood of x intersects some m-parameter family of orbits).

Hence the modality of a smooth function f with an isolated critical point at 0 is the modality of its k-jet under the action of the group of k-jets of diffeomorphisms, for k sufficiently large.

Let's consider the space of polynomials $M = C[x_1,\ldots,x_n]$ as a subset of the space of germs of functions $f(x_1,\ldots,x_n)$ at 0. The *normal form* of a given class K of functions is a mapping $\varphi: B \to M$ from a finite dimensional linear parameter space B to the space of polynomials, satisfying:

1. $\varphi(B)$ intersects all the orbits belonging to K,
2. the corresponding counterimage of every orbit in B is a finite set,
3. the counterimage of the whole complement of K lies in some hypersurface in B.

The normal form is simple, if φ is of the form:

$$\varphi(b_1,\ldots,b_r) = g + b_1x_1' + \cdots + b_rx_r',$$

for $b_i \in C$ and monomials x_i' of M.

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2. CLASSIFICACION OF THE RESIDUAL PARTS.

2.1 DEFINITION. A monomial $x_1^{i_1}\cdots x_n^{i_n}$ has *generalized degree* d for $r=(r_1,\dots,r_n)\in Q^n$ if $r_1 i_1+\cdots+r_n i_n = d$. The *generalized degree* of 0 is ∞.

2.2 DEFINITION. Let $\mathfrak{m}$ be the maximal ideal of the ring $C[[x_1,\dots,x_n]]$. A formal power series $f\in\mathfrak{m}\subset C[[x_1,\dots,x_n]]$ is *quasihomogeneous of type* $(d;r_1,\dots,r_n)$ $(d\in Q^x)$ if each monomial of f has generalized degree d or ∞ for $(r_1,\dots,r_n)$. We call d the *generalized degree* of f and the r_i the *weights* of f.

2.3 DEFINITION. A formal power series $f\in\mathfrak{m}\subset C[[x_1,\dots,x_n]]$ is *non-degenerate* if there exists a natural number n such that $\mathfrak{m}\supset\Delta f\supset\mathfrak{m}^n$, where $\Delta f=(\partial f/\partial x_1,\dots,\partial f/\partial x_n)$ is the ideal spanned by the partial derivatives of f.

If f is a non-degenerate quasihomogeneous formal power series, the local ring $R_f = C[[x_1,\dots,x_n]]/\Delta f$ is a finite dimensional vector space, and we may define the multiplicity μ of the critical point 0 of f as the dimension of R_f. Moreover, the number of basis monomials of the local ring R_f with given generalized degree (for fixed weights $(r_1,\dots,r_n)=r$) is the same for all non-degenerate quasihomogeneous formal power series of type $(d;r)$ (V. I. Arnold [1]).

2.4 DEFINITION. (V. I. Arnold [1]). Let f be a non-degenerate quasihomogeneous series of type $(d;r)$. We call the number of basis monomials of generalized degree $\geqslant d$ the *inner modality* of f and denote it by $m(f)$.

Some years after this definition, B. Matin and G. Pfister proved that the modality is equal to the inner modality if f is a non-degenerate quasihomogeneous polynomial (see [7]).

K. Saito showed in [11] that every non-degenerate quasihomogeneous polynomial with inner modality 1 can be deformed into one of the families P_8, X_9 and J_{10}. These families of singularities are exactly the families with inner modality equal to 1 and type $(d;r)$ for which at least one basis monomial has generalized degree equal to d. This result means that P_8, X_9 and J_{10} are boundaries of non-degenerate quasihomogeneous polynomials with inner modality 0 in a certain sense, and motivated the following:

2.5 DEFINITION. (E. Yoshinaga and M. Suzuki, [12]). A non-degenerate quasihomogeneous polynomial f is a *boundary* of non-degenerate quasihomogeneous polynomials with inner modality k if $m(f) = k + 1$ and the number of basis monomials with generalized degree $> d$ is less than $k + 1$ (f is of type $(d;r)$, $r \in Q^n$). Then, by definition, $b(f) = k$.

In paragraph 5 we will study some of the adjacentnesses of normal forms with inner modality equal to 6, which will make clearer the meaning of the boundaries.

2.6 THEOREM. (V. I. Arnold, [1]). *Let* f *be a non-degenerate quasihomogeneous series of type* $(d;r)$, $r \in Q^n$ *let* N *be the common denominator of the weights* r_i *and write* $r_i = A_i/N$ *Then we have*

$$\chi_f(Z) := \Sigma \mu_i Z^i = \prod_{i=1}^{n} \frac{Z^{N-A_i} - 1}{Z^{A_i} - 1}$$

where $\chi_f(Z)$ *is the Poincaré series of the graded module* R_f, *i.e., the series in the variable* Z *with coefficient* μ_i *equal to the number of basis monomials of* R_r *with generalized degree* i/N. □

By Theorem 2.6 the coefficients of χ_f are symmetric: $\mu_j = \mu_{D-j}$ for $D = nN - 2\Sigma A_i$ and $0 \leqslant j \leqslant D$. Therefore,

(1) $m(f) = \sum_{j \geqslant N} \mu_j = \sum_{j \leqslant D-N} \mu_j$ and if we set $h = D/N$. h is also the highest generalized degree of the basis monomials of R_f.

2.7 THEOREM. (K. Saito, [9]). *Let* $f \in \mathfrak{m} \subset C[[x_1,\dots,x_n]]$ *be a non-degenerate quasihomogeneous series of type* $(d;r)$. *Then there exists a coordinate system* $(y_1,\dots,y_k)$ *such that* $f = h(y_1,\dots,y_k) + y_{k+1}^2 + \cdots + y_n^2$, *where* $h \in C[y_1,\dots,y_k]$ *is a quasihomogeneous polynomial of type* $(1;s_1,\dots,s_k)$, *with* $0 < s_i < ½$ *for* $i = 1,\dots,k$. *The natural number* k *and* $(s_1,\dots,s_k)$ *are uniquely determined up to permutations of components. We call* k *the corank of* f *and* h *the residual part of* f. □

2.8 THEOREM. (V. I. Arnold, [1]). *If two non-degenerate convergent power series*

$$f = \psi(x_1,\dots,x_k) + x_{k+1}^2 + \cdots + x_n^2 \in C\{x_1,\dots,x_n\}$$

$$g = \varphi(x_1,\dots,x_k) + x_{k+1}^2 + \cdots + x_n^2 \in C\{x_1,\dots,x_n\}$$

where $f,g \in \mathfrak{m}$, *are analytically equivalent, then* ψ *and* φ *are so, too. Conversely, if* ψ *and* φ *are analytically equivalent, then* f *and* g *are so.* □

Therefore, we may consider only non-degenerate quasihomogeneous polynomials of type $(1;r)$ with $0 < r_i < 1/2$, and for the classification of non-degenerate quasihomogeneous power series it is sufficient to classify their residual parts.

2.9 THEOREM. (E. Yoshinaga and M. Suzuki, [12]). *Let* f *be a non-degenerate quasihomogeneous polynomial of type* $(1;r)$, $r \in Q^n$. *If* $\Sigma r_i \geq (2n-3)/4$ *(or equivalently, if* $h-1 \leq 1/2$*) and* $j \leq Nh-N$, *then* $\mu_j = \#\{(k_1,\dots,k_n \in N^n \mid \Sigma k_iA_i = j\}$, *where* $\#B$ *denotes the cardinality of the set* B. *Hence, if* $\Sigma r_i \geq (2n-3)/4$, *we have*

$$(2) \qquad m(f) = \#\{(k_1,\dots,k_n) \in N^n \mid \Sigma k_1A_i \leq h-1\}. \qquad \square$$

In order to obtain the new classes of non-degenerate quasihomogeneous functions with inner modality 6, we generate the systems of weights of quasihomogeneous polynomials of type $(1;r_1,\dots,r_n)$ and in each case the inner modality is computed by formula (1) or by formula (2), which is easier to evaluate, but not as general as (1).

3. MAIN THEOREMS.

3.1 THEOREM. *The residual parts of non-degenerate quasihomogeneous polynomials* f *with inner modality* 6 *are exhausted, up to stable equivalence, by table* I:

TABLE I

Notation	Normal form
E_{42}	x^3+y^{22}
E_{43}	x^3+xy^{15}
E_{44}	x^3+y^{23}
J_{40}	$x^3+ax^2y^7+y^{21}$, $4a^3+27 \neq 0$

Z_{39}	$x^3y + ax^2y^7 + y^{19}$, $4a^3 + 27 \neq 0$
Z_{41}	$x^3y + y^{20}$
Z_{42}	$x^3y + xy^{14}$
Z_{43}	$x^3y + y^{21}$
N_{25}	$x^5y + ax^4y^2 + bx^3y^3 + cx^2y^4 + xy^5$, $\Delta(a,b,c) \neq 0$
N^*_{28}	$x^4y + ax^2y^5 + bx^3y^3 + y^9$, $\Delta(a,b) \neq 0$
N_{29}	$x^5y + y^7$
N^1_{30}	$x^5y + xy^6$
N^2_{30}	$x^6 + y^7$
N^1_{31}	$x^4y + y^{10}$
N^2_{31}	$x^6y + y^6$
N^3_{31}	$x^5 + xy^7$
N^1_{32}	$x^4y + xy^8$
N^2_{32}	$x^5 + y^9$
Q_{38}	$x^3 + yz^2 + xy^{12} + ax^2y^6$, $a \neq 0$
Q_{40}	$x^3 + yz^2 + y^{19}$
Q_{41}	$x^3 + yz^2 + xy^{13}$
U_{26}	$x^3 + xz^2 + xy^5 + ay^5z$, $a \neq 0$
U_{28}	$x^3 + xz^2 + y^8$
V^*_{25}	$x^2y + y^4 + z^6 + ay^2z^3$, $a \neq 0$
V^{*1}_{27}	$x^2y + y^6z + z^4 + ay^4z^2 + by^2z^3$, $\Delta(a,b) \neq 0$
V^*_{30}	$x^2y + z^4 + axy^5 + by^9$, $\Delta(a,b) \neq 0$
V^{*2}_{27}	$x^2y + y^4 + xz^4$
V^*_{31}	$x^2y + z^4 + y^7z$
V'^1_{26}	$x^3 + y^3z + xz^4 + ax^2y^2$, $4a^3 + 27 \neq 0$
V'^2_{26}	$x^3 + y^4 + yz^4$
V'_{40}	$ax^{10}z + bxz^2 + cx^{19} + dy^3$, $\Delta(a,b,c,d) \neq 0$

O_{22}^{2}	$x + xy + axy + yz + zw,\ \Delta(a) \neq 0$
O_{24}^{1}	$x^2z + y^3 + ay^2z + z^3 + w^4,\ 4a^3 + 27 \neq 0$
O_{24}^{2}	$xw^2 + yz^2 + y^2w + x^2z$
O_{24}^{3}	$ax^5 + bxz^2 + cx^3w + dy^3 + ey^2z + fyz^2 + gz^3,\ \Delta(a,b,c,d,e,f,g) \neq 0$
O_{24}^{4}	$x^3y + xw^2 + y^2w + z^3$
O_{25}	$x^3z + xw^2 + y^2w + z^3$

The classes J_{40}, Z_{39}, N_{25}, N_{28}^{*}, Q_{38}, U_{26}, V_{25}^{*}, V_{27}^{*1}, $V_{26}^{\prime 1}$, O_{22}^{2} and O_{24}^{1} are also boundaries of non-degenerate quasihomogeneous polynomials with inner modality 5 and were obtained by the authors in [5].

3.2 THEOREM. *The residual parts of boundaries of non-degenerate quasihomogeneous polynomials with inner modality* 6 *are exhausted, up to stable equivalence, by the following table* II:

TABLE II

Notation	Normal Form
J_{46}	$x^3 + ax^2y^8 + y^{24},\ 4a^3 + 27 \neq 0$
X_{33}	$x^4 + ax^3y^3 + by^{12} + x^2y^6,\ a^2 \neq 4,\ b \neq 0$
Z_{45}	$x^3y + ax^2y^8 + y^{22},\ 4a^3 + 27 \neq 0$
N_{34}	$x^4y + ax^2y^6 + y^{11},\ a \neq 4$
Q_{44}	$x^3 + yz^2 + xy^{12}$
S_{30}^{*}	$ax^{11} + bx^8y + cx^6z + dx^5y^2 + ex^3yz + fx^3y^3 + gxz^2 + hy^2z,$ $\Delta(a,b,c,d,e,f,g,h) \neq 0$
V_{24}^{*}	$ax^5 + bx^4y + cx^3y^2 + dx^2y^3 + ex^2yz + fxy^4 + gxy^2z + hxz^2 + iy^5 + jy^3z +$ $+ kyz^2,\ \Delta(a,b,c,d,e,f,g,h,i,j,k) \neq 0$
V_{27}^{*3}	$ax^7 + bx^5y + cx^3y^2 + dx^2z^2 + exy^3 + fyz^2,\ \Delta(a,b,c,d,e,f) \neq 0$
V_{33}^{*}	$ax^{10} + bx^5y^2 + cx^3yz + dxz^2 + ey^4,\ \Delta(a,b,c,d,e) \neq 0$
O_{26}	$ax^5 + bx^3w + cx^2y^2 + dxw^2 + eyz^2 + fy^2w,\ \Delta(a,b,c,d,e,f) \neq 0$

In the above theorems Δ is a polynomial in the coefficients of the normal form and $\Delta \neq 0$ is the condition under which a normal form has an isolated critical point at 0.

4. PROOFS OF THE THEOREMS 3.1 AND 3.2.

4.1 LEMMA. (K. Saito, [9]). *Let* f *be a non-degenerate quasihomogeneous polynomial with* corank $(f) = n$ *and weights* $(r_1,\dots,r_n)$, *then*

(3) $\Sigma r_i \leqslant n/3$ *or equivalently* $h-1 \geqslant (n-3)/3$ □

4.2 LEMMA. *Let* f *be a non-degenerate quasihomogeneous polynomial. If* $m(f) = 6$ *or* $b(6) = 6$, *then* corank $(f) \leqslant 5$.

Proof. Let f be a non-degenerate quasihomogeneous polynomial. $\{1,x_1,\dots,x_n\}$ is always contained in a basis of monomials of R_f. If $n > 5$, by (3) $h-1 \geqslant (n-3)/3 > 2/3 > 1/2$ and from (1) we have $m(f) \geqslant n+1 > 6$ or $b(f) \geqslant n+1 > 6$. □

Therefore, in order to classify the non-degenerate quasihomogeneous polynomials with $m(f) = 6$ or $b(f) = 6$ it is sufficient to consider only those polynomials with corank $(f) \leqslant 5$.

Let f be a non-degenerate quasihomogeneous polynomial. From Theorem 1.4 of [12], for every i, f contains the monomial x_i^m for some integer $m > 0$ or f contains the monomial $x_i^m x_j$ for some integer $m > 0$ and $1 \leqslant j \leqslant n$. Thus if we want to obtain all the systems of quasihomogeneous weights $(r_1,\dots,r_n)$, we can compute them from the systems of n monomials of these two types. The systems of this kind correspond one-to-one to the mappings of the set $E_n = \{x_1,\dots,x_n\}$ to itself: $\{x_1^m x_{i_1}, x_2^m x_{i_2},\dots,x_n^{m_n} x_{i_n}\} \rightsquigarrow g$, where $g\colon E_n \to E_n$ is defined by $g(x_k) = x_{i_k}$ so for the computation of the systems of quasihomogeneous weights it is necessary to consider as equivalent those mappings which become equal after a permutation of the variables, i.e., $g_1 \sim g_2$ iff for any $y \in E_n$, $g_1(y) = \alpha g_2 \alpha(y)$, where α is some element of S_{E_n}, the symmetric group of permutations of E_n.

Let's first consider the problem of counting the equivalence classes of mappings of E_n to itself.

4.3 PROPOSITION. *The number of equivalence classes of mappings of* E_n *to itself is given by the formula*

$$\sum_{(k_1,\dots,k_n)} \prod_{i=1}^{n} \frac{1}{k_i!} \left(\frac{\Theta_i}{i}\right)^{k_i},$$

where $\Theta_i = \sum_{j|i} jk_j$ *and the sum runs over all solutions of* $k_1 + 2k_2 + \cdots + nk_n = n$, *for* $k_i \in \{0,1,\dots,n\}$.

Proof. The equivalence classes of the mappings of E_n are the orbits of S_{E_n} acting of $F_n = \{g \,/\, g\colon E_n \to E_n\}$. By Burnside's lemma [4], the number of equivalence classes is equal to

$$(*) \qquad \frac{1}{|S_n|} \sum_{\alpha \in S_n} \Theta(\alpha),$$

where $\Theta(\alpha)$ is the number of mappings which are equivalent with respect to α, i.e., the mappings g such that $g(x) = \alpha g \alpha(x)$, for every x.

Let α be a permutation whose decomposition in cycles contains k_i cycles ($k_i \in \{0,1,\dots,n\}$) of length i and g be an α-invariant mapping. If $(x_1,\dots,x_r)$ is a cycle of α, $g(x_i) = \alpha g \alpha(x_i) = \alpha g(x_{i+1 \bmod (r)})$, i.e., $g(x_r) \overset{\alpha}{\leftrightarrow} g(x_{r-1}) \overset{\alpha}{\leftrightarrow} \cdots \overset{\alpha}{\leftrightarrow} g(x_1) \overset{\alpha}{\leftrightarrow} g(x_r)$.

Therefore, the values of the sequence $g(x_r)$, which are different among themselves form a cycle $(y_1,\dots,y_p)$ of α, with $p \,|\, r$.

Hence the image of the cycle $(x_1,\dots,x_n)$ by an α-invariant mapping g can be chosen in (**) $\sum_{j|i} jk_j$ ways. The number of permutations whose decomposition in cycles contains k cycles of length i is $n! \prod_{i=1}^{n} \frac{1}{k_i!} (i)^{-k_i}$, so, from (*) and (**), we get the desired formula. □

The first values of the above formula are 3, 7, 19, 47, 130, 343, 951 for n = 2, 3, 4, 5, 6, 7, 8. So we can check the number of classes of quasihomogeneous weights in [1] and [12] for $n \leqslant 4$, and for $n = 5$ we have now that the number of systems of weights is 47.

If we describe the case x_i^m by the figure x_i  and the case $x_i^m x_j$ by $x_j \cdot\!\to\!\cdot\, x_i$, the systems of monomials can be represented by graphs.

For n = corank (f) ≤ 4, we can find in [1] and [12] the corresponding systems of quasihomogeneous weights and associated graphs.

Using the same notation as in Theorem 2.6, the following proposition summarizes the systems of quasihomogeneous weights and associated graphs for corank (f) = 5. Here a, b, c, d and e denote the exponents greater than 1 corresponding to the variables x, y, z, t and w in the system of monomials represented by the graphs below. For example, the graph 12 represents the system x^a, y^b, xz^c, $w^d t$ and wt^e and the associated system of weights is $(\frac{1}{a}, \frac{1}{b}, \frac{a-1}{ac}, \frac{e-1}{de-1}, \frac{d-1}{de-1})$.

4.4 PROPOSITION. *Every non-degenerate quasihomogeneous polynomial of five variables of corank* 5 *(up to isomorphism) contains at least one of the fourty seven systems of monomials in table* III *with non-zero coefficients.*

The following proposition provides us with upper bounds for the non-negative integers a, b, c, d and e in the formulas of the systems of quasihomogeneous weights:

4.5 PROPOSITION. *Let* f *be a non-degenerate quasihomogeneous polynomial with* m(f) = 6 *or* b(f) = 6.

a) *If* corank (f) = 2 *and* (a,b) *is the system of non-negative integers corresponding to* f *([12], Proposition* 3.3*), then* max {a,b} $\leq$ 24.

b) *If* corank (f) = 3 *and* (a,b,c) *is the system of non-negative integers corresponding to* f *([12], Proposition* 3.4*), then* max {a,b,c} $\leq$ 24.

c) *If* corank (f) = 4 *and* (a,b,c,d) *is the system of non-negative integers corresponding to* f *([12], Proposition* 3.5*), then* max {a,b,c,d} $\leq$ 18.

d) *If* corank (f) = 5 *and* (a,b,c,d,e) *is the system of non-negative integers corresponding to* f *in Proposition* 4.4, *then* max {a,b,c,d,e} $\leq$ 4.

Proof. Let f be a non-degenerate quasihomogeneous polynomial of type $(1, r_1, \ldots, r_n)$, let $r_i = A_i/N$ and let $h := n - 2\ r_i$ be the highest generalized degree in the generalized degree of the basis of monomials of R_f (see Theorem 2.6). Then, by Lemma 3.2 of [12]

$$\Sigma r_i \geq (2n-3)/4 \Rightarrow m(f) = \# \{k \in N^n / \Sigma k_i r_i \leq h - 1\}.$$

TABLE III

	A_1	A_2	A_3	A_4	A_5	N
1	bcde	acde	abde	abce	abcd	abcde
2	bcde	acde	abde	abce	(a-1)bcd	abcde
3	bcde	acde	abde	(a-1)bce	(a-1)bcd	abcde
4	bcde	acde	abde	(a-1)bce	(b-1)acd	abcde
5	bcde	acde	abde	(a-1)bce	(ad-a+1)bc	abcde
6	(de-1)bc	(de-1)ac	(de-1)ab	(e-1)abc	(d-1)abc	(de-1)abc
7	bcde	acde	(a-1)bde	(a-1)bce	(b-1)acd	abcde
8	bcde	acde	(a-1)bde	(a-1)bce	(ac-a+1)bd	abcde
9	bcde	acde	(a-1)bde	(b-1)ace	(ac-a+1)bd	abcde
10	bcde	acde	(a-1)bde	(ac-a+1)be	(ac-a+1)bd	abcde
11	bcde	acde	(a-1)bde	(ac-a+1)be	(acd-ac+a-1)b	abcde
12	(de-1)bc	(de-1)ac	(a-1)(de-1)b	(e-1)abc	(d-1)abc	(de-1)abc
13	(cd-1)be	(cd-1)ae	(d-1)abe	(c-1)abe	(cd-d)ab	(cd-1)abe
14	(cde+1)b	(cde+1)a	(de-e+1)ab	(ce-c+1)ab	(cd-d+1)ab	ab(cde+1)
15	bcde	(a-1)cde	(a-1)bde	(ab-a+1)ce	(ab-a+1)cd	abcde
16	bcde	(a-1)cde	(a-1)bde	(ab-a+1)ce	(ac-a+1)bd	abcde
17	bcde	(a-1)cde	(a-1)bde	(ab-a+1)ce	(abd-ab+a-1)c	abcde
18	(de-1)bc	(a-1)(de-1)c	(ab-a+a)(de-1)	(e-1)abc	(d-1)abc	abc(de-1)
19	bcde	(a-1)cde	(ab-a+1)de	(ab-a+1)ce	(abc-ab+a-1)d	abcde
20	bcde	(a-1)cde	(ab-a+1)de	(abc-ab+a-1)e	abcd-abd+ad-d	abcde
21	bcde	(a-1)cde	(ab-a+1)de	(abc-ab+a-1)e	abcd-abc+ab-a+1	abcde
22	(de-1)bc	(a-1)(de-1)c	(ab-a+1)(de-1)	(e-1)abc	(d-1)abc	abc(de-1)
23	(cd-1)be	(a-1)(cd-1)e	(d-1)abc	(c-1)abc	d(a-1)ab	abe(cd-1)
24	b(cde+1)	(a-1)(cde+1)	(de-e+1)ab	(cd-d+1)ab	(cd-c+1)ab	ab(cde+1)

continuation

	A_1	A_2	A_3	A_4	A_5	N
25	(bc-1)de	(c-1)ade	(b-1)ade	c(b-1)ae	b(c-1)ad	ade(bc-1)
26	ade(bc-1)	ade(c-1)	ade(b-1)	ace(b-1)	abcd-abc-ad+ac	ade(bc-1)
27	(bc-1)(de-1)	(ac-1)(de-1)	(ab-1)(de-1)	(ae-a)(bc-1)	(ad-a)(bc-1)	a(bc-1)(de-1)
28	e(bcd+1)	ae(cd-d+1)	ae(bd-b+1)	ac(bc-c+1)	ad(bc-c+1)	ae(bcd+1)
29	bcde-1	acde-ade+ae-a	abde-abe+ab-a	abce-abc+ac-a	abcd-acd+ad-a	a(bcde-1)
30	ac(b-1)	cde(a-1)	bde(a-1)	ace(b-1)	abcd-abd-cd+d	ecd(ab-1)
31	cde(b-1)	cde(a-1)	bde(a-1)	abce-abe-ce+be	abcd-abd-cd+d	cde(ab-1)
32	cde(b-1)	cde(a-1)	bce(a-1)	abce-abe-ce+be	abcd-abc-cd+ab +c-b	cde(ab-1)
33	c(de-1)(b-1)	c(de-1)(a-1)	b(a-1)(de-1)	c(ab-1)(e-1)	c(ab-1)(d-1)	c(ab-1)(de-1)
34	(b-1)(cde+1)	(a-1)(cde+1)	(ab-1)(de-e+1)	(ab-1)(ce-c+1)	(ab-1)(cd-d+1)	(ab-1)(cde+1)
35	de(bc-c+1)	de(ac-a+1)	de(ab-b+1)	ce(ab-b+1)	ad(bc-c+1)	de(abc+1)
36	de(bc-c+1)	de(ac-a+1)	de(ab-b+1)	ce(ab-b+1)	abcd-abc+bc-c+d	de(abc+1)
37	bcde-cde+de-e	acde-ade+ae-e	abde-abe+be-e	abce-bce+ce-e	abcd-bcd+cd-d	e(abcd-1)
38	bcde-cde+de-e+1	acde-ade+ae-a+1	abde-abe+ab-b+1	abce-abc+bc-c+1	abcd-bcd+cd-d+1	abcde+1
39	bcde	(a-1)cde	(a-1)bde	(a-1)bce	(a-1)bcd	abcde
40	bcde	acde	(b-1)ade	ace(b-1)	(b-1)acd	abcde
41	bcde	(a-1)cde	de(ab-a+1)	ce(ab-a+1)	cd(ab-a+1)	acdeb
42	de(bc-1)	ade(c-1)	ade(b-1)	abe(c-1)	abd(c-1)	ade(be-1)
43	de(bc-c+1)	de(ac-a+1)	de(ab-a+1)	be(ac-a+1)	bd(ac-a+1)	de(abc+1)
44	bcde	(a-1)cde	de(ab-a+1)	bce(a-1)	bcd(a-1)	abcde
45	cde(b-1)	ade(c-1)	ade(b-1)	abe(c-1)	abd(c-1)	ade(bc-1)
46	bcde-cde-be+de	ade(c-1)	abe(d-1)	abe(c-1)	abd(c-1)	abe(cd-1)
47	cde(b-1)	cde(a-1)	de(ab-1)	ce(ab-a)	cd(ab-a)	(ab-1)cde

When the above condition holds, using arguments analogous to the case $m(f) = 5$ of $b(f) = 5$ ([5], Lemma 4.7, 4.8 and 4.9), we can prove a), b) and c).

If $\Sigma r_i < (2n-3)/4$, it follows that $m(f) = 6$ or $b(f) = 6$ implies $r_i \geqslant \frac{1}{14-2n}$ (if $r_j < \frac{1}{14-2n}$, then $\{1,x_1,\dots,x_n,x_j^2,\dots,x_j^2\}$ is contained in a basis of R_f, and they all have generalized degree $< h-1$, hence $m(f) \geqslant 7$ or $b(f) \geqslant 7$). The system of non-negative integers $(a,b,c,\dots)$ satisfies the inequalities $1/a \geqslant r_1$, $1/b \geqslant r_2,\dots$ and therefore $a \leqslant 14-2n$, $b \leqslant 14-2n,\dots$

In particular, for $n = 5$ we have from (3) $\Sigma r_i \leqslant 5/3 < 7/4$, and so we get d).

For corank $(f) = n \leqslant 4$, the upper bounds $14-2n$ are smaller than the ones obtained when $\Sigma r_i \geqslant (2n-3)/4$. □

Using a computer, we have computed for $n = 2,3,4,5$ the systems of non-negative integers (a,b), (a,b,c), (a,b,c,d) and (a,b,c,d,e) and the asociated systems of weights corresponding to the Propositions 3.3, 3.4 and 3.5 in [12] and Proposition 4.4 which satisfy the restrictions of Proposition 4.5. For each system of weights the highest generalized degree of the basic monomials is computed and the restriction (3) is imposed. The inner modality is computed by (2) if $h-1 \leqslant 1/2$ and if $h-1 > 1/2$, by (1). Among the systems of integers, there are some which correspond to degenerate polynomials. These degeneracies are shown by the fact that $\Pi(Z^{N-A_i} - 1)$ is not divisible by $\Pi(Z^{A_i} - 1)$ and the Lemmas 3.11 and 3.12 of [12].

For corank $(f) = 5$ we have not obtained new classes of singularities with $m(f) = 6$ or $b(f) = 6$.

5. SOME ADJACENTNESSES.

In this section we give some of the adjacentnesses among the quasihomogeneous functions classified in paragraph 3. In what follows, we use the notation of the normal forms in paragraph 3 for the families of functions $f + \Sigma t_i g_i$, $g_i \in C$, where g_i's are generators of a basis of monomials of R_f with generalized degree > 1.

5.1 DEFINITION. (V. I. Arnold, [2]). For the families K and L, L is *adjacent* to K if every germ of L can be deformed into a germ in K by an arbitrarily small deformation. We denote this by $L \to K$.

The hierarchy induced by the adjacentnesses among the classes of singularities has been studied in many papers (see V. I. Arnold [2], E. Brieskorn [3], E. Yoshinaga and M. Suzuki [12] and the authors [5] and [6]). This subject provides interesting information about the deformation theory of the classes of singularities.

As noticed by K. Saito, the boundaries of non-degenerate quasihomogeneous polynomials seem to play the role of the border between two strata of contiguous modalities. Computing explicitly basis of monomials of the residual parts of the classes of singularities of Theorem 3.1, we have obtained the following diagrams:

$$J_{40} \leftarrow E_{42} \leftarrow E_{43} \leftarrow E_{44}$$

$$Z_{39} \leftarrow Z_{41} \leftarrow Z_{42} \leftarrow Z_{43}$$

$$N_{25} \leftarrow N_{29},\ N^1_{30},\ N^2_{30},\ N^2_{31}$$

$$N^*_{28} \leftarrow N^1_{31},\ N^3_{31},\ N^1_{32},\ N^2_{32}$$

$$\begin{array}{lll} Q_{38} & \leftarrow Q_{40} & \leftarrow Q_{41} \\ & \nwarrow & \\ & V'_{40} & \end{array}$$

$$U_{26} \leftarrow U_{28}$$

$$V^*_{25} \leftarrow V^{*2}_{27}$$

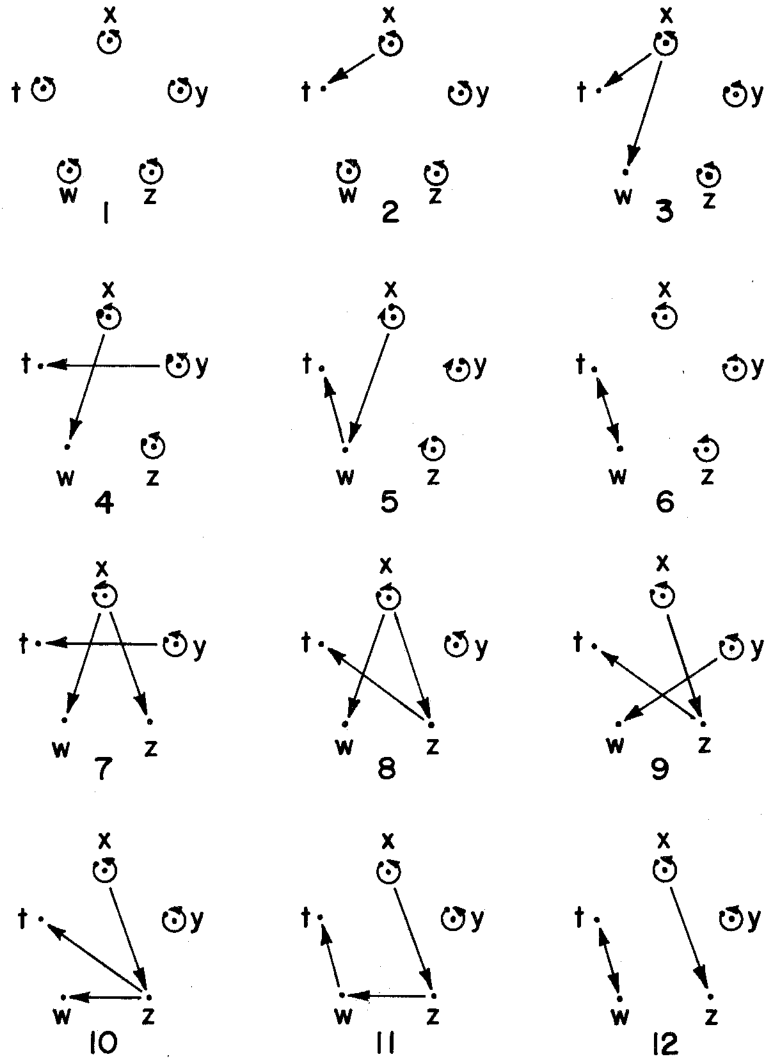
$$V^{*1}_{27} \leftarrow V^*_{30},\ V^*_{31}$$

$$O^2_{22} \leftarrow V'^2_{26}$$

$$O^2_{22} \leftarrow O^2_{24},\ O^3_{24},\ O^4_{24},\ O_{25}$$

Hence, we can extend Theorem 2 in [12] and obtain the following:

5.2 THEOREM. *For* $k \leqslant 5$, *each family of quasihomogeneous functions with inner modality equal to* $k + 1$ *is adjacent to the boundary of quasihomogeneous functions with inner modality equal to* k. □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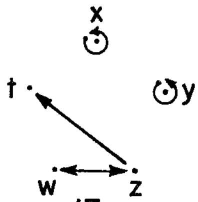

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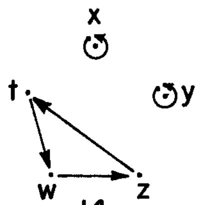

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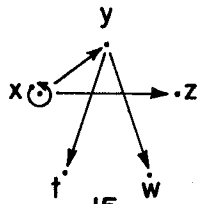

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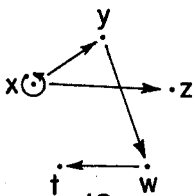

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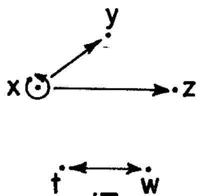

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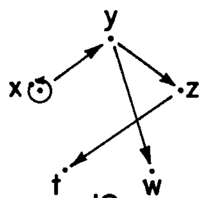

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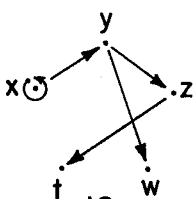

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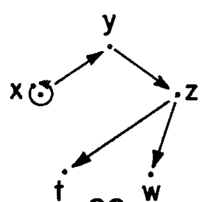

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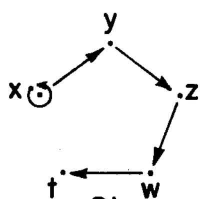

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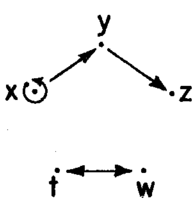

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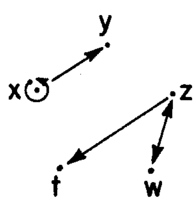

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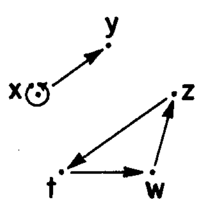

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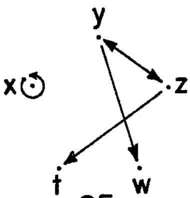

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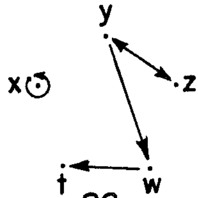

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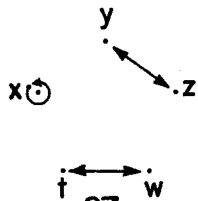

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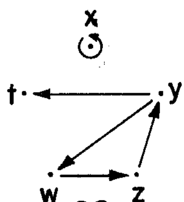

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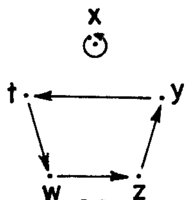

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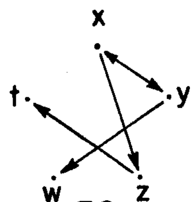

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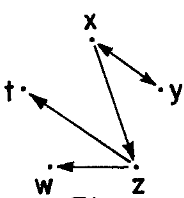

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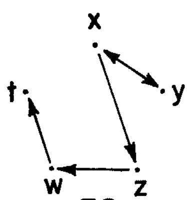

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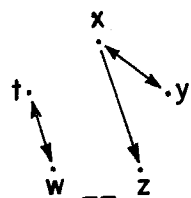

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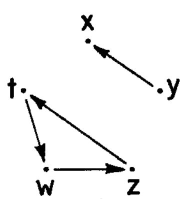

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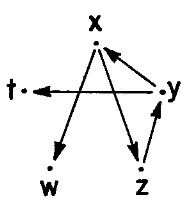

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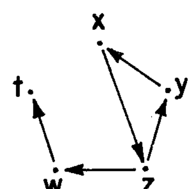

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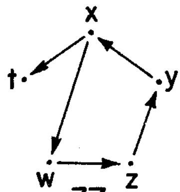

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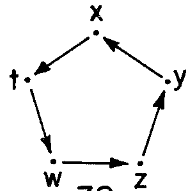

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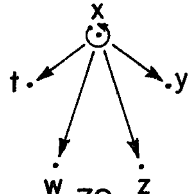

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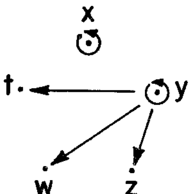

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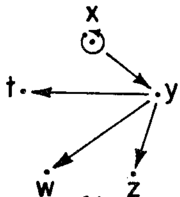

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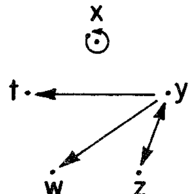

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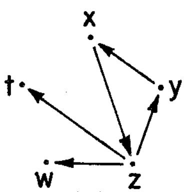

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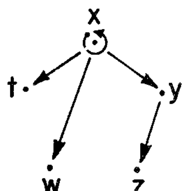

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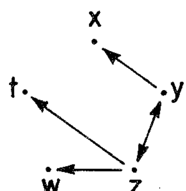

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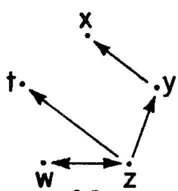

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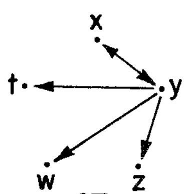

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