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THE THEOREM OF PHILIP HALL FOR VECTOR SPACES

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INTRODUCTION

It is well known the classical theorem of P. Hall which guarantees the necessary and sufficient condition (Hall's condition) for the existence of distinct representatives of set of families. The theorem has been stated in different ways like transversal theory, graph theory, etcetera and in terms just as matchings, transversals, systems of distinct representatives, binary matrices, etcetera (see for reference [1] and [5]). In the present work we will use the term of matching.

Let X and Y be finite sets, $W \subseteq X \times Y$ and for $A \subseteq X$ define the set

$$r_W(A) = r(A) = \{y \in Y \mid \exists \ a \in A \text{ such that } (a, y) \in W\}.$$

Evidently $r(A) \subseteq Y$ and for all $A, B \subseteq X$ the following relations hold:

i) If $A \subseteq B$, then $r(A) \subseteq r(B)$,

ii) $r(A \cap B) \subseteq r(A) \cap r(B)$,

iii) $r(A \cup B) = r(A) \cup r(B)$.

We say that W has a matching if there exists an injective mapping $f: X \to Y$ such that $(x, f(x)) \in W$, for all $x \in X$. Without loss of generality we can suppose that $r(X) = Y$.

We say that W satisfies the so called Hall's condition if

$$|A| \leq |r(A)| \quad \text{for all} \quad A \subseteq X,$$

where $|\cdot|$ denotes the set cardinality. Let us note that if W satisfies the Hall's condition, then the canonical projection $W \to X$ is surjective.

THEOREM (P. Hall). *Let X and Y be finite sets and $W \subseteq X \times Y$. Then W has a matching if and only if W satisfies the Hall's condition.*

The theorem has been proved in many different ways, for a very interesting proof and more details see [1].

This work is the beginning of researches leading to state theorems of Hall's type using more complex structures than sets, for example algebraic structures like vector spaces. Consider two finite dimensional vector spaces V_1 and V_2 over an arbitrary field K. Let G be a vector subspace of the direct sum $V_1 \oplus V_2$ denoted by $G \subseteq_s V_1 \oplus V_2$. Let $\langle a_1, a_2, \ldots, a_k \rangle$ be the vector subspace generated by the vectors $a_1, a_2, \ldots, a_k$ of a vector space. For any subset A of V_1 define

$$r_G(A) = r(A) = \{ b \in V_2 \mid \exists \ a \in A \text{ such that } (a, b) \in G \}.$$

It is easy to show that if $A \subseteq_s V_1$, then $r(A) \subseteq_s V_2$. We say that G has a matching if there exists an injective linear mapping $f: V_1 \to V_2$ such that $(a, f(a)) \in G$ for all $a \in V_1$. We say that G satisfies the Hall's condition if

$$\dim A \leq \dim R(A) \text{ for all } A \subseteq_s V_1,$$

where $R_G(A) = R(A) = \langle r(A \setminus \{0\}) \rangle$. It is clear that $R(A) \subseteq_s r(A)$. As we will see later, this definition of the Hall's condition is natural in the way that it contains implicity the surjectivity of the projection $G \to V_1$. It will be shown that this condition is necessary and suficient for the existence of a matching in G.

Let us recall that the partially ordered set of vector subspaces of a vector space with the order "A is subspace of B" is a modular geometric lattice in which the intersection operation is the set theoretical one and the union is the sum of vector subspaces. The intersection will be denoted by $\wedge$ and the union by $\vee$. Following the usual definition, the direct sum of vector spaces is the disjoint union of them.

1. PRELIMINARY RESULTS

We will assume from now on that V_1, V_2 are finite dimensional vector spaces and $G \subseteq_s V_1 \oplus V_2$.

1. PROPOSITION. *If G satisfies the Hall's condition for vector spaces, then the projection $G \to V_1$ is surjective.*

Proof: Let $a \neq 0$ be a point of V_1 and $A = \langle a \rangle$. We have

$$1 \leq \dim A \leq \dim R(A)$$

and then for some $\lambda \neq 0$ in K there exists b in V_2 such that $(\lambda a, b) \in G$. From it, $(a, b/\lambda) \in G$ is a preimage of a under the canonical projection. ■

2. PROPOSITION. *For all $A, B \subseteq_s V_1$ the following properties are satisfied:*

i) *If* $A \subseteq_s B$, *then* $R(A) \subseteq_s R(B)$,

ii) $R(A \wedge B) \subseteq_s R(A) \wedge R(B)$,

iii) $R(A) \vee R(B) \subseteq_s R(A \vee B)$ *and if Hall's condition holds, then the equality holds.*

Proof: i), ii) and the first part of iii) are trivial. We need just to prove that if G satisfies the Hall's condition, then $R(A \vee B) \subseteq_s R(A) \vee R(B)$.
Let $y \in r((A \vee B) \setminus \{0\})$, and then there exists $a \in A$ and $b \in B$ such that $(a+b, y) \in G$ and $a+b \neq 0$. There are three cases. If $a = 0$ and $b \neq 0$ (symmetrically $a \neq 0$ and $b = 0$), then $y \in R(B) \subseteq_s R(A) \vee R(B)$ (respectively, $y \in R(A) \subseteq_s R(A) \vee R(B)$).
If both $a, b \neq 0$, then by the surjectivity of the projection $G \to V_1$ there is a $(a, y_1) \in G$ with $y_1 \in R(A)$ and

$$(a+b, y) - (a, y_1) = (b, y - y_1) \in G$$

with $y - y_1 \in R(B)$. Therefore

$$y = y_1 + (y - y_1) \in R(A) \vee R(B). \qquad \blacksquare$$

Let us remark that in the same way we did before, one can prove these relations for vector subspaces $r(A)$ of V_2.

3. PROPOSITION. *Let* $A, B \subseteq_s V_1$, $A \wedge B = \{0\}$ *and* $G \wedge (A \oplus V_2)$ *has a matching. If* G *satisfies the Hall's condition, then*

$$r(\{0\}) = r(A) \wedge r(B).$$

Proof: Trivially $r(\{0\}) \subseteq_s r(A) \wedge r(B)$.
Let us fix a basis $\{c_1, c_2, \ldots, c_k\}$ of $r(A) \wedge r(B)$. Let $f\colon A \to V_2$ be a matching of $G \wedge (A \oplus V_2)$. Since f is injective, the preimages

$\{a_1, a_2, \ldots, a_k\}$ by f of the elements of the basis fixed before are linearly independent elements of A. Since $c_i \in r(B)$, there exist $b_i \in B$ such that $(b_i, c_i) \in G$ and since $(a_i, c_i) \in G$, we get $(a_i - b_i, 0) \in G$ for all $i = 1, 2, \ldots, k$. Suppose that $\sum_{i=1}^{k} \lambda_i(a_i - b_i) = 0$, then $\sum_{i=1}^{k} \lambda_i a_i = \sum_{i=1}^{k} \lambda_i b_i$.

Since $\sum_{i=1}^{k} \lambda_i a_i = \sum_{i=1}^{k} \lambda_i b_i \in A \wedge B = \{0\}$, and the a_i are linearly independent, we have that all $\lambda_i = 0$ for $i = 1, 2, \ldots, k$. Thus $\{a_1 - b_1, a_2 - b_2, \ldots, a_k - b_k\}$ are linearly independent vectors too. By Hall's condition

$$\dim r(\langle a_1 - b_1, a_2 - b_2, \ldots, a_k - b_k \rangle) \geq k.$$

Thus, there exist linearly independent vectors $d_1, d_2, \ldots, d_k$ of $r(\langle a_1 - b_1, a_2 - b_2, \ldots, a_k - b_k \rangle)$, and linear combinations of $\{a_j - b_j\}_{j=1}^{k}$ such that $\left(\sum_j \alpha_j^{(i)}(a_j - b_j), d_i\right) \in G$ for all $i = 1, 2, \ldots, k$. Since $(a_i - b_i, 0) \in G$, then $(0, d_i) \in G$ and $d_i \in r(\{0\})$ $(i = 1, 2, \ldots, k)$. Hence

$$\dim r(A) \wedge r(B) = k \leq \dim r(\{0\}).$$

■

4. PROPOSITION. *If Hall's condition holds, then $r(\{0\}) \subseteq_s R(A)$ for all $\{0\} \neq A \subseteq_s V_1$.*

Proof: Let $x \in r(\{0\})$ (and then $(0, x) \in G$). Let $0 \neq a \in A$. Since the projection $G \to V_1$ is surjective there exists $y \in R(A)$ such that $(a, y) \in G$. Thus $(a, x + y) \in G$ which implies that $x + y \in R(A)$ since $a \neq 0$. Consequently $x = x + y - y \in R(A)$. ■

5. COROLLARY. *Under conditions of* Propositions 3 *y* 4 *for any* $B' \neq 0$ *of* B

$$R(A) \wedge R(B) = R(A) \wedge R(B')$$

holds.

Proof: It is sufficient to point out that

$$R(A) \wedge R(B) \subseteq_s r(\{0\}) \subseteq_s R(B')$$

and so

$$R(A) \wedge R(B) \subseteq_s R(A) \wedge R(B').$$

The other inclusion is evident. ■

2. THE MAIN RESULT

1. THEOREM (P. Hall's theorem for vector spaces). *Let* V_1 *and* V_2 *finite dimensional vector spaces over a field* K *and* $G \subseteq_s V_1 \oplus V_2$. *Then* G *has a matching if and only if* G *satisfies the Hall's condition.*

Proof: Suppose that G has a matching, i.e. there exists an injective linear mapping $f: V_1 \to V_2$ such that $(x, f(x)) \in G$ for all $x \in V_1$. If $A \subseteq_s V_1$, then $f(A) \subseteq_s r(A)$ and

$$f(A \setminus \{0\}) \subseteq \langle r(A \setminus \{0\}) \rangle = R(A).$$

Hence

$$\dim A = \dim f(A) \leq \dim R(A)$$

which proves the necessity of Hall's condition. The proof of the sufficiency proceeds by induction on $n = \dim V_1$. If $n = 1$, then there exists

$a \in V_1$ such that $\langle a \rangle = V_1$. By Hall's condition there exists $0 \neq b \in V_2$ such that $(a, b) \in g$. Since G is a vector subspace, the mapping

$$f: V_1 \ni \lambda a \to \lambda b \in V_2$$

is a matching as required. We assume that the result is established for all vector subspaces $G \subseteq_s V_1' \oplus V_2$ such that $\dim V_1' \leq n-1$. Let G be now a subspace of $V_1 \oplus V_2$ such that $\dim V_1 = n$. We are going to construct an injective linear mapping $f: V_1 \to V_2$ satisfying the required conditions.

Let $\dim V_1 = n > 1$. Let $0 \neq a \in V_1$ and $V_1' \subseteq_s V_1$ be a complement of $\langle a \rangle$ in V_1. By induction's hypothesis, there exists $f': V_1' \to V_2$ matching of $G \wedge (V_1' \oplus V_2)$. Also by Hall's condition, $1 = \dim(\langle a \rangle) \leq \dim R(\langle a \rangle)$, then there exists $0 \neq b$ such that $(a, b) \in G$. Let $\{c_i\}_{i=1}^{n}$ a basis of V_1'.

Case 1: If $b \notin \langle \{f'(c_i)\}_{i=1}^{n-1} \rangle$. Then we define

$$\begin{array}{rcl} f: V_1 = \langle a \rangle \oplus V_1' & \to & V_2 \\ \lambda a + v & \to & \lambda b + f'(v) \end{array}$$

which is a matching.

Case 2: If $R(\langle a \rangle) \subseteq \langle \{f'(c_i)\}_{i=1}^{n-1} \rangle$. Since G satisfies the Hall's condition, $n \leq R(V_1) = R(\langle a, \{c_i\}_{i=1}^{n-1} \rangle)$, then there exists $0 \neq d \in R(V_1) \setminus \langle \{f'(c_i)\}_{i=1}^{n-1} \rangle$ and $v \in V_1'$ such that $(v, d) \in G$ (observe that we can choose $v \in V_1'$, since $R(\langle a \rangle) \subseteq \langle \{f'(c_i)\}_{i=1}^{n-1} \rangle$).
Then $(a + v, d + b) \in G$ with $0 \neq d + b \notin \langle \{f'(c_i)\}_{i=1}^{n-1} \rangle$ and

$$\{a + v, c_1, c_2, \ldots, c_{n-1}\}$$

is a basis of V_1. Then we can define a matching $f: V_1 \to V_2$, similar as in Case 1. ∎

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