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MÉXICO

229

COMPLETE BIPARTITE MULTIGRAPHS CAN BE PARTITIONED INTO COMPLETE GRAPHS MINUS A MATCHING

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FECHA DE RECIBIDO: 2 DE ABRIL DE 1991.

COMPLETE BIPARTITE MULTIGRAPHS CAN BE PARTITIONED INTO COMPLETE GRAPHS MINUS A MATCHING

by

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Abstract. Let be $m=2(n-1)$. Denote $\mathbb{K}_{mm}(n)$ the complete bipartite multigraph in which each edge appears exactly n times. Constructions which give a partition of $\mathbb{K}_{mm}(n)$ in parts isomorphic to $\mathbb{K}_{nn}$ minus a matching are described.

1. Introduction

Let us denote by $\mathbb{K}^*_{nn}$ the complete bipartite graph minus a matching. Up to isomorphism $\mathbb{K}^*_{nn}$ is well defined. Laszlo Pyber [1] told to us the following hypothesis:

For any $2 \le n \in \mathbb{N}$ there is a sufficiently large t such that $\mathbb{K}_{tt}$ can be partitioned in parts isomorphic to $\mathbb{K}^*_{nn}$.

The simple argument that a prime can not divide two consecutive numbers gives that if such t exist then $t=kn(n-1)$ with $k \in \mathbb{N}$.

One way to attempt the construction of the desired partition of $\mathbb{K}_{tt}$ is the following:

Let us group the vertices of $\mathbb{K}_{tt}$ into disjoint classes in such way that any class has exactly n vertices and it is an independent set of vertices, i.e. each class is contained in one of the two parts of $\mathbb{K}_{tt}$. For each two such classes in different parts of $\mathbb{K}_{tt}$ let us delete from it a graph isomorphic to $\mathbb{K}^*_{nn}$ and contained in the subgraph of $\mathbb{K}_{tt}$ induced by the union of the two classes. The remaining graph G consist of matchings of cardinality n between each two classes and if G can be partitioned into $\mathbb{K}^*_{nn}$ then so can $\mathbb{K}_{tt}$. Hence, if we identify in G the vertices of each class between themselves, then the complete bipartite multigraph $\mathbb{K}_{mm}(n)$ is obtained (where $m=k(n-1)$). If G can be partitioned into $\mathbb{K}^*_{nn}$ then so can $\mathbb{K}_{mm}(n)$. However the reciprocal of this last assertion

$\mathbb{Z}_m$. If $\alpha=(a,b)$ is an edge of $\mathbb{K}_{mm}(n)$ then we can talk about the length of α by means of the distance between a and b in $\mathbb{Z}_m$.

Let A be a matching in $\mathbb{K}_{mm}(n)$ i.e. a set of edges pairwise no incident. If $|A|=n$ then A defines in a obvious way a set of edges A^* of a graph isomorphic to $\mathbb{K}^*_{nn}$ Observe that $(A')^*= (A^*)'$, $(A+k)^* = A^*+k$, etc.

Denote

$$\sigma = \{ (i,n-1-i) \mid i \in \{0,1,\dots,n-1\} \}$$
$$\eta = \{ (i,-i) \mid i \in \{0,1,\dots,n-1\} \}$$
$$\tau = \{ (i,-i-1) \mid i \in \{1,\dots,n-1\} , i \text{ odd } \} \cup \{ (i,-i+1) \mid i \in \{1,\dots,n-1\} , i \text{ even } \} \cup \{(0,0)\}$$

The sets σ , η and τ are matchings of cardinality n.

<u>Factorization Theorem for $\mathbb{K}_{mm}(n)$</u>

If n is odd then $\mathbb{K}_{mm}(n)=(\sigma^*+\mathbb{Z}_m) \cup (\tau^*+\mathbb{P}_m) \cup (\tau'^*+\mathbb{I}_m)$.
if n is even then $\mathbb{K}_{mm}(n)=(\sigma^*+\mathbb{Z}_m) \cup (\eta^*+\mathbb{Z}_m)$.

3. The proof.

Denote the length of an edge α by $l(\alpha)$ and let us consider the following function:

$$\varphi(\alpha)=\begin{cases} 0 & \text{if } l(\alpha) \equiv n \mod \mathbb{P}_m \\ 1 & \text{otherwise} \end{cases}$$

i.e. $\varphi(\alpha)=1$ if and only if the parity of $l(\alpha)$ and n are different.

Lemma 1. *An edge of $\mathbb{K}_{mm}$ appears in $\sigma^*+\mathbb{Z}_m$ exactly $n-l(\alpha)-\varphi(\alpha)$ times.*

Proof. We have

$$\sum_{\alpha\in\mathbb{K}_{mm}} (n-l(\alpha)-\varphi(\alpha)) = nm^2 - m^2/2 - 2m \sum_{k=1}^{n-2} k - m^2/2 = nm^2/2$$

Proof. We have

$$\sum_{\alpha\in \mathbb{K}_{mm}} (1+l(\alpha)-odd(\alpha)) = m^2 + m^2/2 + 2m \sum_{k=1}^{n-2} k - m^2/2 = nm^2/2$$

and this is the number of edges in $(\tau^*+\mathbb{P}_m) \cup (\tau'^*+\mathbb{I}_m)$. Hence for proving the lemma it is sufficient to show that each edge appears at least $1+l(\alpha)-odd(\alpha)$ times.

It is easy to see that the translation $x \longrightarrow x+2$ is an automorphism of $(\tau^*+\mathbb{P}_m) \cup (\tau'^*+\mathbb{I}_m)$; therefore it is enough to consider only edges of type $(0,b)$ and $(1,b)$. The edge $(0,b)$ appears in all of the sets of type τ^*-k (k even) and τ'^*+k (k odd) when k is in $\{0,1,\dots,n-1\}$ except in those of them where

1) $b \in \{-k,-k-1,\dots,-k-n+1\}$ and k even or

$b \in \{k,k+1,\dots,k+n-1\}$ and k odd

2) $\left.\begin{array}{l}(0,b) \in \tau'+k \text{ , } k \text{ odd} \\ (0,b) \in \tau-k \text{ , } k \text{ even}\end{array}\right\} \Rightarrow 2k+1=b$

Condition 2) holds exactly once if b is an odd number, i.e. $odd(0,b)$ times. If $b<n-1$, then condition 1) holds when $k \in \{0,1,\dots,n-2-b\}$ and k is even, or $k \in \{b+1,b+2,\dots,n-1\}$ and k is odd, i.e. $\left\lceil\frac{n-b-1}{2}\right\rceil + \left\lfloor\frac{n-b-1}{2}\right\rfloor = n-1-b = n-1-l(0,b)$ times. If $b>n-1$, then condition 1 holds when $k \in \{1-b,2-b,\dots,n-1\}$ and k is even or $k \in \{0,1,\dots,b-n\}$ and k is odd, i.e. $\left\lceil\frac{n+b-1}{2}\right\rceil + \left\lfloor\frac{n+b-1}{2}\right\rfloor = n-1+b = n-1-l(0,b)$ times. If $b=n-1$, then condition 1) never holds. So for edges of type $(0,b)$ the lemma is proved.

Let us deal now with the edges of type $(1,b)$. The edge $(1,b)$ appears in all the sets of type τ^*+n+k with k odd and τ'^*+n-k with k even, and k in $\{0,1,\dots,n-1\}$, except in those of them where

1) $b \in \{-k+1,-k,\dots,-k-n+2\}$ and k even or

$b \in \{k+1,k+2,\dots,k+n\}$ and k odd

2) $(1,b) \in \tau+n+k$,k odd $\Rightarrow 2k=b$

$(1,b) \in \tau'+n-k$,k even $\Rightarrow 2k=-b$

Condition 2) holds once when b is an even number, i.e. $odd(1,b)$ times. If $b\in\{1,2,\dots,n-1\}$ then condition 1) holds when $k \in \{b,b+1,\dots,n-1\}$ and k is odd, or $k \in \{0,\dots,n-1-b\}$ and k is even, i.e. $\left\lceil\frac{n-b}{2}\right\rceil + \left\lfloor\frac{n-b}{2}\right\rfloor = n-b = n-1-l(1,b)$ times. If $b \in \{n+1,$

$n+2,\dots, m-1,0\}$ then condition 1 holds when $k \in \{0,1,\dots,b-n-1\}$ and k is odd or $k \in \{2-b,2-b+1,\dots,n-1\}$ and k is even i.e. $\left\lceil\frac{n-2+b}{2}\right\rceil + \left\lfloor\frac{n-2+b}{2}\right\rfloor = n-b+2 = n-1-l(1,b)$ times. If $b=n$ then this condition never holds and if $b=1$, then it holds $n-2$ times ■

Proof of factorization Theorem. By lemmas 1 and 3 if n is even we have that in $(\sigma^*+\mathbb{Z}_m) \cup (\eta^*+\mathbb{Z}_m)$ each edge appears exactly n times. If n is odd then the same thing holds for $(\sigma^*+\mathbb{Z}_m) \cup (\tau^*+\mathbb{P}_m) \cup (\tau'^*+\square_m)$ by virtue of lemmas 1 and 4 . ■

References

[1] L. Pyber. Personal communication 1989.